

Annexes

In this set of Annexes, either some advanced topics are covered, or some topics are revisited from an alternative point of view, which allows for a better understanding of the topic. In Annex A, a precise mathematical derivation of the PPGP equations is presented. In addition, the relationship between the PPGP equations for a particle and its antiparticle, if both are moving in the same external world, is discussed. In Annex B, the Dirac equation is presented and discussed. Heuristic discussions related to materials presented in Chap. 8 are presented in Annexes C and I. An in-depth discussion of the antiparticle PPGP equations is presented in Annex D. The solution of the complementary PPGP equation for a particle confined in a one-dimensional infinite well is presented in Annex E. Annex F contains a general discussion about the mathematical apparatus of quantum mechanics. The superposition principle and its relationship with the PPGP equations is the topic discussed in Annex G. Finally, Annex H is directed to quantum mechanics instructors. The pedagogical values of the approach followed by the authors in this book are discussed.

Annex A: Schrödinger-Like and Pauli-Like Relativistic Wave Equations

In this Annex, a precise mathematical derivation of the PPGP equations is presented. In addition, the relationship between the PPGP equations for a particle and its antiparticle, if both are moving in the same external world, is discussed.

Theorem I

By solving the following Schrödinger-like equations (Chap. 3):

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2\mu(r)} \nabla^2 \Psi + V(r)\Psi, \text{ with } \mu(r) = \left[1 + \frac{E - V(r)}{2mc^2} \right] m. \quad (\text{A.1})$$

And:

$$i\hbar \frac{\partial}{\partial t} \Omega = -\frac{\hbar^2}{2\mu'(r)} \nabla^2 \Omega + V(r)\Omega, \text{ with } \mu'(r) = \left[-1 + \frac{E' - V(r)}{2mc^2} \right] m. \quad (\text{A.2})$$

We obtain the exact solutions, Ψ and E , and Ω and E' , of the Klein-Gordon (KG) equation with total energy $E_T = E + mc^2$ and $E_T = E' - mc^2$, respectively.

Demonstration:

The KG equation for a spin- $(s = 0)$ particle, with mass m , in the external central potential $V(r)$ is:

$$\left[i\hbar \frac{\partial}{\partial t} - V(r) \right]^2 \psi_{KG} = -\hbar^2 c^2 \nabla^2 \psi_{KG} + m^2 c^4 \psi_{KG}. \quad (\text{A.3})$$

The time independent equations that correspond to Eqs. (A.1)–(A.3) are:

$$-\frac{\hbar^2}{2\mu(r)} \nabla^2 \varphi + V(r)\varphi = E\varphi, \text{ with } \Psi = \varphi e^{-\frac{i}{\hbar} E t}. \quad (\text{A.4})$$

$$-\frac{\hbar^2}{2\mu'(r)} \nabla^2 \chi + V(r)\chi = E'\chi, \text{ with } \Omega = \chi e^{-\frac{i}{\hbar} E' t}. \quad (\text{A.5})$$

$$-\hbar^2 c^2 \nabla^2 \phi = \{ [E_T - V(r)]^2 - m^2 c^4 \} \phi, \text{ with } \psi_{KG} = \phi e^{-\frac{i}{\hbar} E_T t}. \quad (\text{A.6})$$

By substituting E_T by $E + mc^2$ in Eq. (A.6), and after some algebraic manipulations, we obtain:

$$\left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(r) - \frac{[E - V(r)]^2}{2mc^2} \right\} \phi = E\phi. \quad (\text{A.7})$$

But also, Eq. (A.7) can be obtained by pre-multiplying both sides of Eq. (A.4) by $\mu(r)/m$ and substituting $\mu(r)$ by its value in Eq. (A.1). This is the first half of the demonstration.

Similarly, by substituting E_T by $E' - mc^2$ in Eq. (A.6), and after some algebraic manipulations, we obtain:

$$\left\{ \frac{\hbar^2}{2m} \nabla^2 + V(r) + \frac{[E' - V(r)]^2}{2mc^2} \right\} \phi = E'\phi. \quad (\text{A.8})$$

But also, Eq. (A.8) can be obtained by pre-multiplying both sides of Eq. (A.5) by $\mu'(r)/m$ and substituting $\mu'(r)$ by its value in Eq. (A.2). This completes the demonstration.

Theorem II

By solving the following Pauli-like equations (Chap. 6):

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \Psi + V \Psi, \quad V = eA_o. \quad (\text{A.9})$$

And:

$$i\hbar \frac{\partial}{\partial t} \Omega = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu'} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \Omega + V \Omega. \quad (\text{A.10})$$

We can obtain the spinors and energies, Ψ and E , as well as Ω and E' , that correspond to the exact solutions of the Dirac equation and total particle energies $E_T = E + mc^2$ and $E_T = E' - mc^2$, respectively. In Eqs. (A.9) and (A.10), \mathbf{A} and A_o represent the vector and scalar potentials of electromagnetism. Note that the effective masses $\mu(r)$ and $\mu'(r)$ in Eqs. (A.9) and (A.10) are given by their values in Eqs. (A.1) and (A.2), respectively.

Demonstration

It is well-known that the Dirac equation (Annex B) is the correct Lorentz covariant equation that describes the interaction of a spin- $(s = 1/2)$ particle, with mass m and charge e , with an external electromagnetic field. The stationary solutions of the Dirac equation are bi-spinors, which means they possess four components. They are of the form:

$$\Psi_D = \begin{pmatrix} \Psi \\ \Omega \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-i \frac{E_T}{\hbar} t}, \quad \text{with } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \text{and } \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (\text{A.11})$$

It is also widely known that the time independent Dirac equation is equivalent to the following system of two coupled spinor equations (Annex B):

$$c \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \chi = (E_T - mc^2 - eA_o) \varphi, \quad \text{with } \hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \quad (\text{A.12})$$

$$c \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \varphi = (E_T + mc^2 - eA_o) \chi. \quad (\text{A.13})$$

In Eq. (A.12), σ_i are the Pauli matrices. Eq. (A.13) can be rewritten as:

$$\frac{c}{E_T + mc^2 - eA_o} \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \varphi = \chi. \quad (\text{A.14})$$

Substituting χ given by Eq. (A.14) in Eq. (A.12), we obtain:

$$c\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\left[\frac{c}{E_T + mc^2 - eA_o}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi = (E_T - mc^2 - eA_o)\varphi. \quad (\text{A.15})$$

Finally, making $V = eA_o$ and $E_T = E + mc^2$ in Eq. (A.15), we obtain the time independent Pauli-like PPGP equation:

$$\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\left[\frac{1}{2\mu}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi + V\varphi = E\varphi, \text{ with } \frac{1}{2\mu} = \frac{c^2}{E + 2mc^2 - V}. \quad (\text{A.16})$$

Consequently, the time dependent Pauli-like PPGP equation is Eq. (A.9). This is the first half of the demonstration.

Similarly, Eq. (A.12) can be rewritten as:

$$\frac{c}{(E_T - mc^2 - eA_o)}\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\chi = \varphi. \quad (\text{A.17})$$

Substituting φ given by Eq. (A.17) in Eq. (A.13), we obtain:

$$c\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\left[\frac{c}{(E_T - mc^2 - eA_o)}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\chi = (E_T + mc^2 - eA_o)\chi. \quad (\text{A.18})$$

Finally, making $V = eA_o$ and $E_T = E' - mc^2$ in Eq. (A.18), we obtain the complementary time independent Pauli-like PPGP equation:

$$\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\left[\frac{1}{2\mu'}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\chi + V\chi = E'\chi, \text{ with } \frac{1}{2\mu'} = \frac{c^2}{E - 2mc^2 - V}. \quad (\text{A.19})$$

Consequently, the complementary time dependent Pauli-like PPGP equation is Eq. (A.10). This completes the demonstration.

It is worth noting, as stated in Sect. 6.4, that the Dirac wavefunctions given by Eq. (A.11) are bi-spinors. This means that for the total energy of the particle (E_T), Ψ_D includes a spinor φ and a spinor χ . If $E_T = E + mc^2$, Theorem II states that φ can be obtained by solving the Pauli-like PPGP equation (Eq. A.9). After φ is known, we can use Eq. (A.14) for obtaining χ . In contrast, if $E_T = E' - mc^2$, Theorem 2 states that χ can be obtained by solving the complementary Pauli-like PPGP equation (Eq. A.10). After χ is known, we can use Eq. (A.17) for obtaining φ . This topic is covered in more detail in Annex G.

Theorem III

The particle's wave equations that correspond to $E_T = E' - mc^2$, can be obtained from the antiparticle's wave equations that correspond to $E_{Ta} = E_a + mc^2$. Also, the antiparticle's wave equations that correspond to $E_{Ta} = E'_a - mc^2$, can be obtained from the particle's wave equations that correspond to $E_T = E + mc^2$.

Demonstration for Particles with Spin-($s = 0$)

An antiparticle is just another particle. A particle and its antiparticle contain the same mass m but opposite electric charge. Therefore, if the particle and the antiparticle are moving in the same external central potential, then the PPGP equations that correspond to a spin-($s = 0$) charged particle are Eqs. (A.1) and (A.2). But because of the charge difference, the following equations correspond to the antiparticle:

$$i\hbar \frac{\partial}{\partial t} \Psi_a = -\frac{\hbar^2}{2\mu_a(r)} \nabla^2 \Psi_a - V(r) \Psi_a, \text{ with } \mu_a(r) = \left[1 + \frac{E_a + V(r)}{2mc^2} \right] m. \quad (\text{A.20})$$

$$i\hbar \frac{\partial}{\partial t} \Omega_a = -\frac{\hbar^2}{2\mu'_a(r)} \nabla^2 \Omega_a - V(r) \Omega_a, \text{ with } \mu'_a = \left[-1 + \frac{E'_a + V(r)}{2mc^2} \right] m. \quad (\text{A.21})$$

A comparison between Eqs. (A.1) and (A.2) and Eqs. (A.20) and (A.21) reveals that V changes to $-V$. This is because the particle and the antiparticle have charges of opposite signs, but they are moving in the same external world. The complex conjugate of Eq. (A.20) is the following equation:

$$-i\hbar \frac{\partial}{\partial t} \Psi_a^* = -\frac{\hbar^2}{2\mu_a(r)} \nabla^2 \Psi_a^* - V(r) \Psi_a^*. \quad (\text{A.22})$$

Or:

$$i\hbar \frac{\partial}{\partial t} \Psi_a^* = \frac{\hbar^2}{2\mu_a(r)} \nabla^2 \Psi_a^* + V(r) \Psi_a^*. \quad (\text{A.23})$$

But $E' = -E_a$, so therefore:

$$-\mu_a(r) = \left[-1 - \frac{E_a + V(r)}{2mc^2} \right] m = \left[-1 + \frac{E' - V(r)}{2mc^2} \right] m = \mu'(r). \quad (\text{A.24})$$

Substituting $\mu_a(r)$ by $-\mu'$ in Eq. (A.23), we obtain:

$$i\hbar \frac{\partial}{\partial t} \Psi_a^* = -\frac{\hbar^2}{2\mu'(r)} \nabla^2 \Psi_a^* + V(r) \Psi_a^* \quad (\text{A.25})$$

Therefore, (Ψ_a^*) satisfies Eq. (A.2). This means:

$$\Omega = \Psi_a^* \quad (\text{A.26})$$

This is the first half of the demonstration.

Similarly, the complex conjugate of Eq. (A.1) is the following equation:

$$-i\hbar \frac{\partial}{\partial t} \Psi^* = -\frac{\hbar^2}{2\mu(r)} \nabla^2 \Psi^* + V(r) \Psi^* \quad (\text{A.27})$$

Or:

$$i\hbar \frac{\partial}{\partial t} \Psi^* = \frac{\hbar^2}{2\mu(r)} \nabla^2 \Psi^* - V(r) \Psi^* \quad (\text{A.28})$$

But $E'_a = -E$, so therefore:

$$-\mu(r) = \left[-1 - \frac{E - V(r)}{2mc^2} \right] m = \left[-1 + \frac{E'_a + V(r)}{2mc^2} \right] m = \mu'_a(r) \quad (\text{A.29})$$

Substituting $\mu(r)$ by $-\mu'_a$ in Eq. (A.28), we obtain:

$$i\hbar \frac{\partial}{\partial t} \Psi^* = -\frac{\hbar^2}{2\mu'_a(r)} \nabla^2 \Psi^* - V(r) \Psi^* \quad (\text{A.30})$$

Therefore, Ψ^* satisfies Eq. (A.21). This means:

$$\Omega_a = \Psi^* \quad (\text{A.31})$$

This completes the demonstration. It is worth noting that Eqs. (A.26) and (A.31) imply that:

$$|\Omega|^2 = |\Psi_a|^2 \text{ and } |\Omega_a|^2 = |\Psi|^2 \quad (\text{A.32})$$

Demonstration for Particles with Spin-(s = 1/2)

If a particle with mass m and its antiparticle are moving in the same external electromagnetic field, then the PPGP equations that correspond to a spin-(s = 1/2) charged particle are Eqs. (A.9) and (A.10). But because of the charge difference, the following equations correspond to the antiparticle:

$$i\hbar \frac{\partial}{\partial t} \Psi_a = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi_a - V \Psi_a, \text{ with } \mu_a(r) = \left[1 + \frac{E_a + V(r)}{2mc^2} \right] m. \quad (\text{A.33})$$

$$i\hbar \frac{\partial}{\partial t} \Omega_a = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu'_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Omega_a - V \Omega_a, \text{ with } \mu'_a = \left[-1 + \frac{E'_a + V(r)}{2mc^2} \right] m. \quad (\text{A.34})$$

A comparison of Eqs. (A.9) and (A.10) and Eqs. (A.33) and (A.34) reveals that V changes to $-V$ and $-eA/c$ changes to $+eA/c$. This is because the particle and the antiparticle have charges of opposite signs, but they are moving in the same external world. The complex transpose conjugate of Eq. (A.33) is depicted in the following equation:

$$-i\hbar \frac{\partial}{\partial t} \Psi_a^\dagger = \left\{ \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \right\}^\dagger \Psi_a^\dagger - V \Psi_a^\dagger. \quad (\text{A.35})$$

But:

$$\left\{ \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \right\}^\dagger = \hat{\sigma}^\dagger \cdot \left(-\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma}^\dagger \cdot \left(-\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \quad (\text{A.36})$$

The Pauli matrices given by Eq. (6.19) are Hermitic. This means that they are equal to their transpose conjugates:

$$\hat{\sigma}^\dagger = [\hat{\sigma}_x^\dagger, \hat{\sigma}_y^\dagger, \hat{\sigma}_z^\dagger] = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^* = \hat{\sigma}. \quad (\text{A.37})$$

Therefore, utilizing Eqs. (A.36) and (A.37), we can rewrite Eq. (A.35) in the following way:

$$-i\hbar \frac{\partial}{\partial t} \Psi_a^\dagger = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \Psi_a^\dagger - V \Psi_a^\dagger. \quad (\text{A.38})$$

Or:

$$i\hbar \frac{\partial}{\partial t} \Psi_a^\dagger = -\hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu_a} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi_a^\dagger + V \Psi_a^\dagger. \quad (\text{A.39})$$

But $E' = -E_a$. Therefore, utilizing Eq. (A.24) and substituting $\mu_a(r)$ by $-\mu'$ in Eq. (A.39), we obtain:

$$i\hbar \frac{\partial}{\partial t} \Psi_a^\dagger = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu'(r)} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi_a^\dagger + V \Psi_a^\dagger \quad (\text{A.40})$$

Therefore, (Ψ_a^\dagger) satisfies Eq. (A.10). This means:

$$\Omega = \Psi_a^\dagger. \quad (\text{A.41})$$

This is the first half of the demonstration.

Similarly, the complex transposed conjugate of Eq. (A.9) is the following equation:

$$-i\hbar \frac{\partial}{\partial t} \Psi^\dagger = \left\{ \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \right\}^\dagger \Psi^\dagger + V \Psi^\dagger. \quad (\text{A.42})$$

But:

$$\left\{ \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \right\}^\dagger = \hat{\sigma}^\dagger \cdot \left(-\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma}^\dagger \cdot \left(-\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right). \quad (\text{A.43})$$

Therefore, utilizing Eqs. (A.43) and (A.37), we can rewrite Eq. (A.42) in the following way:

$$-i\hbar \frac{\partial}{\partial t} \Psi^\dagger = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi^\dagger + V \Psi^\dagger. \quad (\text{A.44})$$

Or:

$$i\hbar \frac{\partial}{\partial t} \Psi^\dagger = -\hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi^\dagger - V \Psi^\dagger \quad (\text{A.45})$$

But $E'_a = -E$. Therefore, utilizing Eq. (A.29) and substituting $\mu_a(\mathbf{r})$ by $-\mu'$ in Eq. (A.45), we obtain:

$$i\hbar \frac{\partial}{\partial t} \Psi^\dagger = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu'_a(r)} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi^\dagger - V \Psi^\dagger \quad (\text{A.46})$$

Therefore, (Ψ^\dagger) satisfies Eq. (A.34). This means:

$$\Omega_a = \Psi^\dagger. \quad (\text{A.47})$$

This completes the demonstration. It is worth noting that Eqs. (A.41) and (A.47) imply that:

$$|\Omega|^2 = |\Psi_a^\dagger|^2 \text{ and } |\Omega_a|^2 = |\Psi^\dagger|^2. \quad (\text{A.48})$$

The meaning of Eq. (A.48) could be illustrated with some instances. As shown in Fig. 8.3, the electron in a Hydrogen-like atom has bound quantum states where the electron's total energy is $E_T = E + mc^2$. These electronic bound states are spatially localized around the nucleus of the Hydrogen-like atom. As illustrated in Fig. 8.5, due to Eq. (A.48), the repulsive interaction of a positron with the nucleus of a Hydrogen-like atom produces bound quantum states where the positron's total energy is E_{Ta}

$= E'_a - mc^2$. These positronic bound states are also spatially localized around the nucleus of the Hydrogen-like atom.

In contrast, as shown in Fig. 8.4, the repulsive interaction of a positron with the nucleus of a Hydrogen-like atom produces unbound quantum states where the positron's total energy is $E_{Ta} = E_a + mc^2$. These positronic unbound states are spatially delocalized. As illustrated in Figs. 8.3–8.5, due to Eq. (A.48), the attractive interaction of the electron with the nucleus of a Hydrogen-like atom produces unbound quantum states where the electron's total energy is $E_T = E' - mc^2$. These electronic unbound states are also spatially delocalized.

Annex B: Dirac Equation

The Dirac equation for a free particle with spin- $(s = 1/2)$, charge e and mass m , that is interacting with an external electromagnetic field is:

$$i\hbar \frac{\partial}{\partial t} \Psi_D = c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \Psi_D + eA_o \Psi_D + mc^2 \hat{\beta} \Psi_D. \quad (\text{B.1})$$

In Eq. (B.1), \mathbf{A} and A_o are the vector and scalar potentials of electromagnetism ($V = eA_o$). Ψ_D is the 4-components wavefunction given by Eq. (A.11). The operator $\hat{\beta}$ and each component of the vectorial operator $\hat{\alpha}$ are 4×4 matrices:

$$\hat{\beta} = \begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix}, \text{ with } \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.2})$$

$$\hat{\alpha} = (\hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z), \text{ with } \hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}, i = x, y, z. \quad (\text{B.3})$$

In Eq. (B.2), the x , y , and z components of the operator $\hat{\alpha}$ are defined in terms of the respective Pauli matrix given by Eq. (6.19). Looking for stationary solutions of the Dirac equation of the form:

$$\Psi_D = \begin{pmatrix} \Psi \\ \Omega \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-i \frac{E_T}{\hbar} t}, \text{ with } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \text{ and } \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (\text{B.4})$$

Substituting Eq. (B.4) in (B.1), we obtain:

$$E_T \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + eA_o \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + mc^2 \hat{\beta} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \quad (\text{B.5})$$

Utilizing Eq. (B.2), we can rewrite Eq. (B.5) in the following way:

$$c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = [E_T - eA_o] \begin{pmatrix} \varphi \\ \chi \end{pmatrix} - mc^2 \begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \quad (\text{B.6})$$

But:

$$mc^2 \begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = mc^2 \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}. \quad (\text{B.7})$$

This allows to rewrite Eq. (B.6) as:

$$c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = [E_T - eA_o] \begin{pmatrix} \varphi \\ \chi \end{pmatrix} - mc^2 \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}. \quad (\text{B.8})$$

Utilizing Eq. (B.3), we obtain:

$$c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \left[\sum_{i=x}^z \hat{a}_i \right] \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \text{with } \hat{a}_i = \begin{bmatrix} 0 & c\hat{\sigma}_i \left(\hat{p}_i + \frac{e}{c} A_i \right) \\ c\hat{\sigma}_i \left(\hat{p}_i + \frac{e}{c} A_i \right) & 0 \end{bmatrix} \quad (\text{B.9})$$

Or:

$$c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \chi \\ \varphi \end{pmatrix}. \quad (\text{B.10})$$

Utilizing Eq. (B.10), we can rewrite Eq. (B.8) in the following way:

$$c\hat{\alpha} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = [E_T - eA_o] \begin{pmatrix} \chi \\ \varphi \end{pmatrix} - mc^2 \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}. \quad (\text{B.11})$$

This is equivalent to the system of two spinor equations (Eqs. A.12 and A.13):

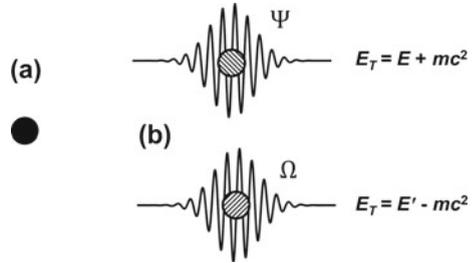
$$\hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \chi = [E_T - eA_o - mc^2] \varphi. \quad (\text{B.12})$$

$$\hat{\sigma} \cdot \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \varphi = [E_T - eA_o + mc^2] \chi. \quad (\text{B.13})$$

Annex C: Classical Versus Quantum

As discussed in Sect. 2.7, the observed stability of the atoms is thoroughly explained

Fig. C.1 **a** There are no waves associated to a classical particle, **b** but two waves associated to a relativistic quantum particle



in non-relativistic quantum mechanics. This is accomplished by assuming that there is a wave associated with a quantum particle. This wave is described by a wavefunction that ends up as a solution of a wave equation. As discussed in Chap. 2, if the particle has spin- $(s = 0)$, the wave equation is the Schrödinger equation. However, as discussed in Chap. 6, if the particle has spin- $(s = 1/2)$, the wave equation is the Pauli equation.

The introduction of the special theory of relativity in quantum mechanics produces fundamental consequences. One of them is that there is not a wave, but there are two waves associated to a relativistic quantum particle. In contrast with nonrelativistic quantum particles, relativistic quantum particles can exist in two kinds of quantum states.

The total energy of the relativistic quantum particle is $E_T = E + mc^2$ when the particle is in the first kind of quantum state. These states are associated with wave functions that are solutions of a wave equation, which is directly related to a nonrelativistic wave equation. If the relativistic quantum particle has spin- $(s = 0)$, the wave equation relates to the Schrödinger-like PPGP equation (Eq. A.1). In the nonrelativistic limit, Eq. (A.1) coincides with Schrödinger equation. However, if the relativistic quantum particle has spin- $(s = 1/2)$, the wave equation is related to the Pauli-like PPGP equation (Eq. A.9). In the nonrelativistic limit, Eq. (A.9) coincides with Pauli equation. The PPGP equations can be formally obtained from the Schrödinger and Pauli equations by substituting the mass of the particle (m) by its relativistic effective mass μ , which is given by Eq. (A.1). This facilitates the simultaneous study of nonrelativistic and relativistic quantum mechanics.

The total energy of the relativistic quantum particle is $E_T = E' - mc^2$ when the particle is in in the second kind of quantum states. We call these states “exotic” because they do not exist in nonrelativistic quantum mechanics. Exotic states are associated with complementary wave equations. If the relativistic quantum particle has spin- $(s = 0)$, the complementary wave equation is the complementary Schrödinger-like PPGP equation (Eq. A.2). However, if the relativistic quantum particle has spin- $(s = 1/2)$, the complementary wave equation is the complementary Pauli-like PPGP equation (Eq. A.10). The relativistic effective mass of the quantum particle in the complementary wave equations is not μ , but μ' given by Eq. (A.2).

As illustrated in Fig. C.1, the most important and notable difference between a classical and a relativistic quantum particle is that there is not any wave associated with a classical particle, but there are two waves associated with a relativistic quantum particle. It is worth noting that there is never a single wave associated with a relativistic quantum particle. For instance, Eqs. (B.12) and (B.13) are valid for the electron, which is a spin- $(s = 1/2)$ quantum particle. Therefore, χ does not exist if φ does not exist, and vice versa.

Antiparticles are particles (other particles). Therefore, everything that we have discussed above about relativistic quantum particles is also valid for relativistic quantum antiparticles. There are two waves associated with a relativistic quantum antiparticle. However, an antiparticle without waves associated to it is then considered to be a classical antiparticle. We could then argue that something transcendental may occur if a relativistic quantum particle or antiparticle loses for any reason a wave, and then both waves associated to it.

Interestingly, as it will be discussed below, there are good reasons to believe that relativistic quantum particles could get “naked” in extreme physical conditions.

Atoms Cannot Be Too Heavy

As discussed in Sect. 8.1, atoms with $Z > 137$ do not exist. There is an attractive Coulombic interaction between the electron and the nucleus in a Hydrogen-like atom. This interaction produces bound states where the total energy of the electron is $E_T = E + mc^2$. As sketched in Fig. 8.3, the electrons' wavefunctions in these bound states (Ψ) are spatially localized around the nucleus. As shown in Fig. 8.1, the spatial localization of Ψ increases when Z increases. Eventually, Ψ collapses to a point when Z is too large. When this happens, the relativistic quantum particle loses the two waves associated to it (Ψ and Ω). The “naked” particle without waves cannot form a stable atom. As discussed in Sect. 2.7, this is a fundamental result of relativistic quantum mechanics. Consequently, there are no atoms with $Z > 137$.

As shown in Fig. 8.1, the collapse of Ψ only occurs if the electron in the Hydrogen-like atom is theoretically described using relativistic quantum mechanics. This indicates that the inexistence of atoms with $Z > 137$ could be considered as an observational fact supporting the validity of both special theory of relativity and quantum mechanics.

Quantum Objects Cannot Be Too Massive

Large bodies with $m > m_p \approx 22 \mu\text{g}$ are classical objects. This is a hypothesis presented in Sect. 8.2. This hypothesis argues that there should be a gravitational attractive interaction between different regions in a massive body. If this large body were a quantum object, then two waves (Ψ and Ω) should be associated to it. Consequently, the hypothetical internal interaction should produce spatially localized quantum states (Ψ), where the total energy of the quantum object is $E_T = E + mc^2$. As shown in Fig. 8.2, the spatial localization of Ψ increases when m increases. Eventually, when m is too large, Ψ collapses to a point. When this happens, the relativistic quantum object loses the two waves associated to it (Ψ and Ω). The “naked” body without waves cannot be a quantum object, but a classical one.

As shown in Fig. 8.2, the collapse of Ψ only occurs if the quantum object is theoretically described using relativistic quantum mechanics. It seems like we are often surrounded by macroscopic bodies that have an apparent classical nature. Our everyday experiences could then be considered as an enormous volume of observational facts supporting (1) the validity of the proposed hypothesis (a quantum object should gravitationally interact with itself), (2) special theory of relativity, and 3) quantum mechanics.

We are not Surrounded by Antimatter

Antimatter does not surround us, which is an observational fact. It is also a huge mystery because current physical theories predict that we should be surrounded by antimatter, but we are not. Nevertheless, an interesting hypothesis about this is discussed in Sect. 8.3. This hypothesis states that charged antimatter bodies containing $|q| > q_P \approx 11e$ should be classical objects. In contrast, charged matter objects with any electrical charge should be quantum objects. The hypothesis places the rupture of the theoretical symmetry between matter and antimatter in the electrostatic interaction of a charged relativistic quantum body with itself. An antimatter charged relativistic quantum object should interact electrostatically with itself in a different manner than how a matter charged relativistic quantum body interacts electrostatically with itself.

The hypothesis argues that there should be an electrostatic repulsive interaction between different regions in an extended antimatter object with charge q . If this antimatter body were a quantum object, then two waves (Ψ_a and Ω_a) should be associated to it. Consequently, the hypothetical internal interaction should produce spatially localized “exotic” quantum states (Ω_a), where the total energy of the antimatter object is $E_{Ta} = E'_a - mc^2$.

We should note that the argument revolves around the spatial localization of exotic antimatter states Ω_a . These states do not exist in nonrelativistic quantum mechanics. Therefore, this argument implies the theoretical framework of relativistic quantum mechanics.

The hypothesis argues that the spatial localization of the wavefunction Ω_a is produced by a repulsive interaction. This may be perceived with some skepticism, but this is because we have argued that the spatial localization of the wavefunction Ψ is produced by an attractive interaction. Certainly, as sketched in Fig. 8.4, the repulsive Coulomb interaction between a positron and the nucleus of a Hydrogen-like atom produces the spatial delocalization of the positronic wavefunction Ψ_a . However, as sketched in Fig. 8.5, the same repulsive interaction produces the spatial localization of the exotic positronic wavefunction Ω_a .

In relativistic quantum mechanics, attractive external interactions may produce the spatial localization of Ψ and Ψ_a , but the spatial delocalization of the exotic states Ω and Ω_a . In contrast, repulsive external interactions may produce the spatial localization of the exotic states Ω and Ω_a , but the spatial delocalization of Ψ and Ψ_a . The equations of relativistic quantum mechanics do not include a possible interaction of a particle or antiparticle with itself. However, the hypothesis that we are considering explores the possibility of such interactions.

As shown in Fig. 8.7, the spatial localization of the exotic state Ω_a increases when $|q|$ increases. Eventually, when $|q|$ is too large, Ω_a collapses to a point. When this happens, the antimatter quantum object loses the two waves associated to it (Ω_a and Ψ_a). The “naked” antimatter body without waves cannot be a quantum object, but a classical one.

We should emphasize and elaborate on how the theoretical symmetry between matter and antimatter is broken in this hypothesis. The internal repulsive Coulomb interaction spatially delocalizes both the matter wavefunction Ψ and the antimatter wavefunction Ψ_a . However, this repulsive interaction spatially localizes the antimatter exotic wavefunction Ω_a , but spatially delocalizes the matter wavefunction Ω . This hypothesis does not explain the reason Mother Nature behaves in this way. However, it correctly predicts that Mother Nature should act as it seems to be.

The rest of the argument is straightforward. Antimatter atoms with $Z \geq 12$ cannot exist because a cloud formed by 12 positrons would be a “naked” antimatter object without waves, which is a classical object. Charged classical antimatter cannot form antimatter atoms. Antimatter atoms are needed for the existence of antimatter living beings. We are surrounded by matter, but not by the existence of antimatter. This is because of how the condition “classical if $|q| > q_P \approx 11e$ ” does not apply to matter.

As stated above, the collapse to a point of Ω_a only occurs if the antimatter object is theoretically described using relativistic quantum mechanics. We are not surrounded by antimatter. This everyday experience could then be considered as a humongous aggregate of observational facts supporting the validity of (1) the proposed hypothesis (a charged quantum object should electrically interact with itself), (2) special theory of relativity, and (3) quantum mechanics.

Primordial Black Holes and Antimatter Electrical Sinks

Primordial black holes with a relatively small mass may exist. Primordial black holes may have been created around 13 billion years ago, at the beginning of our universe. Mass fluctuations with $m > m_P$ could have produced their formation. As discussed above, these hypothetical mass fluctuations may have formed primordial relativistic quantum objects. If their masses were larger than Planck’s mass, then the collapse to a point of Ψ may have created primordial black holes. At present, there is no observational evidence of the existence of primordial black holes. Nevertheless, the possible existence of these small mass black holes is a research topic of great interest in the modern day.

Similarly, primordial antimatter electric sinks may have been created around 13 billion years ago, at the beginning of our universe. Antimatter charge fluctuations with $|q| > q_P$ could have produced their formation. As discussed above, these hypothetical antimatter charge fluctuations may have formed primordial relativistic quantum objects. If their charges were larger than Planck’s charge, then the collapse to a point of Ω_a may have created primordial electric sinks. This may explain the existence of an excess of charged matter in the rest of the universe.

Annex D: Antiparticles

As discussed in Sects. 3.3 and 4.3, an antiparticle is another particle. Historically, the existence of electrons was widely accepted before positrons were discovered in a cloud chamber in 1932. It was later proposed that there exists a relationship between positrons and the theoretically predicted Dirac's quantum states, where the electron total energy is $E_T = E' - mc^2$. This relationship is illustrated in Fig. 3.3. As mentioned in Sect. 3.2, it was initially difficult to accept the existence of quantum states where the kinetic energy of a particle is negative. This motivated Schrödinger to put forth his undivided attention on a nonrelativistic wave equation.

Today, we know that electrons and positrons are always created in pairs. Which one we call the particle, the electron, or the positron, is just an historical accident. Electrons are commonly referred to as particles because they were discovered before the positron. However, it would be strictly correct to refer to positrons as particles and electrons as the antiparticles associated to them.

It is worth noting that if the hypothesis discussed in the previous Annex were to be correct, then we should solely be surrounded by matter. This would transform the historical accident into a necessity. Electrons are not antiparticles, but particles because we are surrounded and formed by them. The positron is the antiparticle of the electron, so positrons must be scarce. This is why they were discovered later. We could differentiate a charged particle from a charged antiparticle because we are not surrounded and formed by antiparticles, but by only particles instead.

Nevertheless, we will only consider the interaction of particles and antiparticles with the external world from now on. In this framework, relativistic quantum mechanics predict the existence of a complete matter-antimatter symmetry.

Today, we know that there is a relationship between the "exotic" particle's states (Ω), where $E_T = E' - mc^2$, and the antiparticle's states (Ψ_a) where $E_{Ta} = E_a + mc^2$. As expected from the theoretical matter-antimatter symmetry, there is also a relationship between the exotic antiparticle's states (Ω_a), where $E_{Ta} = E'_a - mc^2$, and the particle's states (Ψ) where $E_T = E + mc^2$. These relationships are stated in Theorem III in Annex A.

This means that given a common external world, if we know E_a and Ψ_a , we could obtain E' and Ω in the following way (Eqs. A.26 and A.41):

$$E' = -E_a, \text{ and } \begin{cases} \Omega = \Psi_a^* & \text{if } s = 0 \\ \Omega = \Psi_a^\dagger & \text{if } s = 1/2 \end{cases} \quad (\text{D.1})$$

Also, if we know E and Ψ , we could obtain E'_a and Ω_a in the following way (Eqs. A.31 and A.47):

$$E'_a = -E, \text{ and } \begin{cases} \Omega_a = \Psi^* & \text{if } s = 0 \\ \Omega_a = \Psi^\dagger & \text{if } s = 1/2 \end{cases} \quad (\text{D.2})$$

Therefore, given a common external world, we only need to solve the PPGP equations that correspond to the particle and antiparticle of “regular” quantum states. If $s = 0$, these equations are (Eqs. A.1 and A.20):

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2\mu(r)}\nabla^2\Psi + V(r)\Psi, \text{ with } \mu(r) = \left[1 + \frac{E - V(r)}{2mc^2}\right]m. \quad (\text{D.3})$$

And:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = -\frac{\hbar^2}{2\mu_a(r)}\nabla^2\Psi_a - V(r)\Psi_a, \text{ with } \mu_a(r) = \left[1 + \frac{E_a + V(r)}{2mc^2}\right]m. \quad (\text{D.4})$$

If $s = 1/2$, these equations are (Eqs. A.9 and A.33):

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\left[\frac{1}{2\mu}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\Psi + V\Psi, V = eA_o. \quad (\text{D.5})$$

And:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \hat{\sigma}\cdot\left(\hat{\mathbf{p}} + \frac{e}{c}\mathbf{A}\right)\left[\frac{1}{2\mu_a}\right]\hat{\sigma}\cdot\left(\hat{\mathbf{p}} + \frac{e}{c}\mathbf{A}\right)\Psi_a - V\Psi_a. \quad (\text{D.6})$$

All these equations are Schrödinger-like or Pauli-like equations. This makes it possible to simultaneously learn relativistic and nonrelativistic quantum mechanics.

Annex E: A Relativistic Quantum Particle Confined in a Small Spatial Region

As discussed in Chap. 2, the simplest model for explaining the stability of atoms is a quantum particle confined in a one-dimensional infinite well. This model captures the essential properties of a quantum particle, including that it has a wave associated to it. This wave is described by a wavefunction that can be obtained by solving a wave equation. The spatial localization of the wave associated to the quantum particle results in a discrete set of possible energy values. The minimum amount of energy possible is $E = K > 0$. Consequently, like an electron in the Hydrogen atom, the spatially confined quantum particle cannot lose all its kinetic energy.

In Sect. 3.6, the model was improved from the consideration that the wave equation that should be solved was not the Schrödinger equation, but the Schrödinger-like PPGP equation instead. For pedagogical simplicity, some naïve suppositions were introduced in Sect. 3.6. First, like in nonrelativistic quantum mechanics, it was assumed that the wavefunction of the relativistic quantum particle should be null outside of the infinite well. This naïve assumption was addressed in Chap. 4. It was

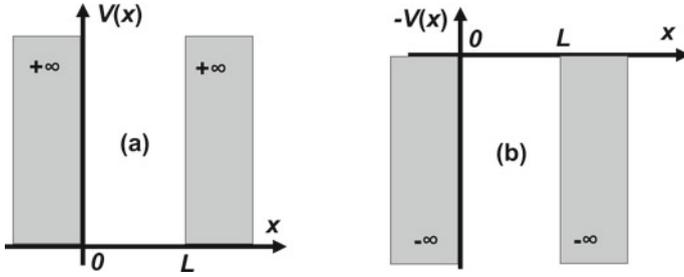


Fig. E.1 A charged spin- $(s = 0)$ particle **a** and **b** its antiparticle, that interact with the same external world, experience potentials with opposite signs

discussed that, in contrast with the mass of the particle, the effective mass of a relativistic quantum particle (μ) is not constant. The Klein paradox was discussed, and it occurs because $\mu < 0$ if $V > 2mc^2$.

Second, like in nonrelativistic quantum mechanics, it was assumed that there is a single wavefunction (Ψ) associated to a relativistic quantum particle. Consequently, the total energy of the particle was assumed to be $E_T = E + mc^2$. We know now that there also is a second wavefunction (Ω) associated to a relativistic quantum particle. The total energy of the particle in these exotic states is not $E_T = E + mc^2$, but instead $E_T = E' - mc^2$. In these exotic states the kinetic energy of the particle is negative.

For completing the relativistic quantum mechanics description of a charged particle (with spin $s = 0$) in a one-dimensional infinite well, we should solve the complementary Schrödinger-like PPGP equation (Eq. A.2):

$$i\hbar \frac{\partial}{\partial t} \Omega = -\frac{\hbar^2}{2\mu'(x)} \nabla^2 \Omega + V(x)\Omega, \quad \text{with } \mu'(x) = \left[-1 + \frac{E' - V(x)}{2mc^2} \right] m. \quad (\text{E.1})$$

Alternatively, if we assume a charged antiparticle that experiences the same external world than the charged particle, we should solve the Schrödinger-like PPGP equation for the antiparticle (Eq. A.20):

$$i\hbar \frac{\partial}{\partial t} \Psi_a = -\frac{\hbar^2}{2\mu_a(x)} \nabla^2 \Psi_a - V(x)\Psi_a, \quad \text{with } \mu_a(x) = \left[1 + \frac{E_a + V(x)}{2mc^2} \right] m. \quad (\text{E.2})$$

The potentials $V(x)$ and $-V(x)$ are sketched in Fig. E.1a, b, respectively. The energies (E_a) of the antiparticle in the state equal to Ψ_a is $E_a > -V(x)$ everywhere. In addition, the antiparticle's relativistic effective mass is $\mu_a > 0$ everywhere. Therefore, there are no bound solutions of Eq. (E.2). Consequently, a continuous energy spectrum is associated to the antiparticle. Utilizing Eq. (D.1), we can then predict that there are no bound solutions of Eq. (E.1). There is a continuous amount of possible energy values, $E' < 0$, if the particle is in an exotic state Ω . Therefore, in the exotic

states Ω , the kinetic energy of the relativistic quantum particle is negative inside of the infinite well.

The energy distribution of the electronic states in the Hydrogen atom is sketched in Figs. 8.3a and 8.5c. Unsurprisingly, if we retain the naïve assumption that Ψ is null outside of the infinite well, then the energy distribution of particle's states in the infinite one-dimensional well is also like the one sketched in these figures. Like the Coulomb potential in the Hydrogen atom, $V(x)$, corresponds to an attractive interaction of the particle with the external world. This attractive external interaction spatially localizes ψ but delocalizes Ω . Particle's Ψ states are bound, but exotic particle's Ω states are unbound. There is then a discrete set of energy values, $E_T = E + mc^2$, but a continuous amount of energy values, $E_T = E' - mc^2$.

As sketched in Fig. 8.4a, to an empty exotic particle state Ω corresponds the energy $E' < 0$. Thus in Eq. (D.1), $E' < V(x)$ and $\mu' < 0$ everywhere. This corresponds to an unbound hole in the Dirac's Sea. The relativistic quantum antiparticle tends to sink to the Ψ_a state with minimum positive energy (Fig. 8.4b), while a hole (an empty exotic particle's state) tends to float in the Dirac's Sea to the exotic particle's state Ω with maximum negative energy (Fig. 8.4a).

As sketched in Fig. 8.5, a particle is the antiparticle of the antiparticle. The antiparticle's wavefunctions Ψ_a can be obtained by solving Eq. (E.2). As discussed above, the solutions of Eq. (E.2) correspond to unbound antiparticle's states and a continuous set of energy values, $E_{Ta} = E_a + mc^2$. The antiparticle's wavefunctions Ω_a can be obtained by solving the following equation (Eq. A.21):

$$i\hbar \frac{\partial}{\partial t} \Omega_a = -\frac{\hbar^2}{2\mu'_a(r)} \nabla^2 \Omega_a - V(x) \Omega_a, \text{ with } \mu'_a = \left[-1 + \frac{E'_a + V(x)}{2mc^2} \right] m. \quad (\text{E.3})$$

The potential $-V(x)$ is sketched in Fig. E.1b. We do not need to solve Eq. (E.3) because we already know the solution of the particle wave equation (Eq. A.1):

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2\mu(r)} \nabla^2 \Psi + V(r) \Psi, \text{ with } \mu(r) = \left[1 + \frac{E - V(r)}{2mc^2} \right] m. \quad (\text{E.4})$$

In Sect. 3.6, assuming that Ψ is null outside of the infinite well, Eq. (E.4) was solved. It was found that Eq. (E.4) has bound solutions (Ψ) and there is a discrete set of energies, $E_T = E + mc^2$. The energy that corresponds to a bound state Ψ is $E > 0$; therefore, $E > V(x)$ inside of the well but $E < V(x)$ outside of it.

Due to Eq. (D.2), Eq. (E.3) has bound solutions ($\Omega_a = \Psi^*$) and there is a discrete set of energies, $E_T = E'_a - mc^2$. In contrast with the particle states, the energy that corresponds to a bound exotic antiparticle state Ω_a is $E'_a = -E < 0$. Therefore, $E'_a < V(x)$ inside of the well, but $E'_a > V(x)$ outside of it. As sketched in Fig. 8.5b, an empty exotic antiparticle state Ω_a corresponds to a bound hole in the Dirac's Sea. The relativistic quantum particle tends to sink to the Ψ state with minimum positive energy (Fig. 8.5c), while a hole (an empty exotic antiparticle's state) tends to float in the Dirac's Sea to the exotic antiparticle state Ω_a with maximum negative energy (Fig. 8.5b).

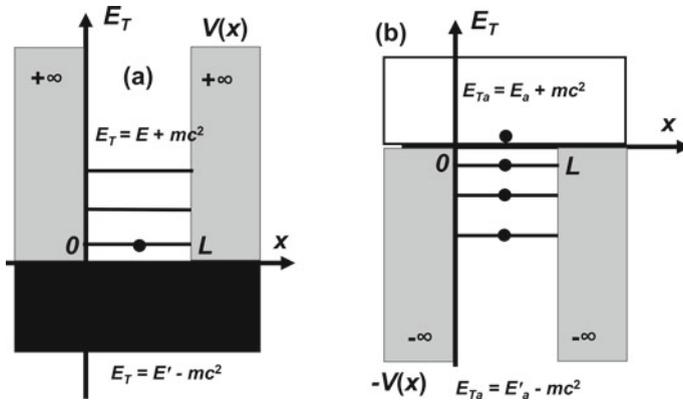


Fig. E.2 Distribution of energy states that correspond to a **a** charged spin- $(s = 0)$ particle and **b** its antiparticle moving in the same external world. The external world confines the particle in an infinite well

The energy distribution of the positronic states, which are a result of the repulsive interaction of a positron with the nucleus of a Hydrogen atom, is sketched in Fig. 8.5b. Foreseeable, if we retain the naïve assumption that Ψ is null outside of the infinite well, and if the antiparticle interacts with the same infinite one-dimensional well that traps the particle, then the energy distribution of antiparticle's states is also like the one sketched in Fig. 8.5b. Like the repulsive Coulomb interaction between a positron and the nucleus of a Hydrogen atom, $-V(x)$ corresponds to a repulsive interaction of the antiparticle with the external world. This repulsive external interaction spatially delocalizes Ψ_a but localizes the exotic antiparticle states of Ω_a . Antiparticle's Ψ_a states are unbound, but exotic antiparticle's Ω_a states are bound. There is then a continuous set of energy values, $E_{Ta} = E_a + mc^2$, but discrete set of energy values, $E_{Ta} = E'_a - mc^2$.

The discussion above is summarized in Fig. E.2. As sketched in Fig. E.2a, the attractive interaction of a charged particle with the potential $V(x)$, which is produced by the external world that surrounds the particle, spatially localizes the particle states Ψ . This produces a discrete set of energy values, $E_T = E + mc^2$. The same external attractive interaction spatially delocalizes the exotic particle states Ω . This produces a continuous set of energy values, $E'_T = E' - mc^2$. An observable particle trapped in the infinite well corresponds to all the exotic particle states Ω being occupied by particles. In addition, the observed particle occupies the bound particle state Ψ with minimum energy in the ground state.

As sketched in Fig. E.2b, the repulsive interaction of the charged antiparticle with the same external world that surrounds the particle, spatially delocalizes the antiparticle states Ψ_a . This produces a continuous set of energy values, $E_{Ta} = E_a + mc^2$. The same external repulsive interaction spatially localizes the exotic antiparticle states Ω_a . This produces a discrete set of energy values, $E'_{Ta} = E'_a - mc^2$. An observable antiparticle, that is in the same infinite well that traps the particle, corresponds to

all the exotic antiparticle states Ω_a being occupied by antiparticles. In addition, the observed antiparticle occupies an unbound antiparticle state (Ψ_a).

Annex F: Mathematical Formalism of Quantum Mechanics

In this Annex, without being extremely careful about the mathematical intricacies involved in this subject, we want to give the reader a general idea about the mathematical apparatus of quantum mechanics. For simplicity, we will refer to nonrelativistic quantum mechanics where there is a wave associated with a quantum particle. This wave is mathematically represented by a wavefunction. Wavefunctions are solutions of linear wave equations and are multi-dimensional vectors of a Hilbert space. Several mathematical topics should be extensively covered to give a practical meaning to the previous sentences.

Linear Algebra in Quantum Mechanics

[In quantum mechanics, a vector space consists of a set of vectors (wavefunctions $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$), together with a set of scalars (complex numbers a, b, c, \dots) which are subject to two operations of vector addition and scalar multiplication].

The vector addition satisfies the following properties. The sum of any two vectors is another vector:

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle. \quad (\text{F.1})$$

The vector addition is commutative:

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle. \quad (\text{F.2})$$

And associative:

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle. \quad (\text{F.3})$$

There exist a zero (or null) vector, $|0\rangle$, with the property that for every vector $|\alpha\rangle$:

$$|\alpha\rangle + |0\rangle = |\alpha\rangle. \quad (\text{F.4})$$

And for every vector $|\alpha\rangle$, there is an associated inverse vector $|\alpha^{-1}\rangle$, such that:

$$|\alpha\rangle + |\alpha^{-1}\rangle = |0\rangle. \quad (\text{F.5})$$

The scalar multiplication satisfies the following properties. The product of any scalar with any vector is another vector:

$$a|\alpha\rangle = |\beta\rangle. \quad (\text{F.6})$$

Scalar multiplication is distributive with respect to vector addition:

$$a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle. \quad (\text{F.7})$$

Also, scalar multiplication is distributive with respect to scalar addition:

$$(a + b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle. \quad (\text{F.8})$$

It is also associative with respect to the ordinary multiplication of scalars:

$$a(b|\alpha\rangle) = (ab)|\alpha\rangle. \quad (\text{F.9})$$

From these properties it follows that:

$$0|\alpha\rangle = |0\rangle \text{ and } |-\alpha\rangle = (-1)|\alpha\rangle. \quad (\text{F.10})$$

[Always remember that, in quantum mechanics, wavefunctions are vectors of a linear algebra and the scalars are complex numbers].

A linear combination of the vectors $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$ is an expression of the form:

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \dots \quad (\text{F.11})$$

A vector $|\lambda\rangle$ is said to be linearly independent of the set of vectors $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$ if it cannot be written as a linear combination of these vectors. By definition, a set of vectors is linearly independent if each one is linearly independent of all the rest.

[For example, any polynomial equation can be represented as a linear combination of other polynomials. If we choose our set of linearly independent vectors to include all real polynomials, $\{\phi\} = a + bx + cx^2 + \dots$, then the equation $f(x) = x + 3x^2$ can be formed using coefficients $a = 0, b = 1, c = 3$ and all other coefficients equal to 0. This is precisely what it means to be a linear combination. Note that the components of this linear combination cannot be represented as a linear combination themselves. This means x cannot be written as an addition (combination) of x^2 or x^3 and so on].

A collection of vectors is said to span the “space formed by all vectors” if every vector can be written as a linear combination of the members of this set. A set of linearly independent vectors that span a space is considered to be a basis. The number of vectors in any basis is called the dimension of the space.

[For instance, the wavefunctions given by Eq. (2.54) in Chap. 2 (which are solutions of the time independent Schrödinger equation for a one-dimensional infinite well), are the vectors:

$$|\varphi_n\rangle = \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \text{ with } n = 1, 2, \dots \quad (\text{F.12})$$

These vectors form a basis of the space formed by all the wavefunctions $\phi(x)$ in the form:

$$|\phi\rangle = \phi(x) = \sum_{n=1}^{+\infty} c_n \varphi_n(x). \quad (\text{F.13})$$

The dimension of this space of functions is infinite because there are infinite linearly independent functions $\varphi_n(x)$.

For the moment, let us assume that the dimension of a space (N) is finite. With respect to a prescribed basis:

$$|\theta_1\rangle, |\theta_2\rangle, \dots, |\theta_N\rangle. \quad (\text{F.14})$$

Any given vector:

$$|\alpha\rangle = \sum_{n=1}^N a_n |\theta_n\rangle. \quad (\text{F.15})$$

It is uniquely represented by the ordered N -tuple of its components:

$$|\alpha\rangle \leftrightarrow (a_1, a_2, \dots, a_N). \quad (\text{F.16})$$

This is of practical importance because it is often easier to work with the component than with the abstract vectors themselves. For instance, to add vectors, we add their corresponding components:

$$|\alpha\rangle + |\beta\rangle \leftrightarrow (a_1 + b_1, a_2 + b_2, \dots, a_N + b_N). \quad (\text{F.17})$$

To multiply by a scalar, you multiply both components:

$$c|\alpha\rangle \leftrightarrow (ca_1, ca_2, \dots, ca_N). \quad (\text{F.18})$$

The null vector is represented by a string of zeros:

$$|0\rangle \leftrightarrow (0, 0, \dots, 0) \quad (\text{F.19})$$

And:

$$|-\alpha\rangle \leftrightarrow (-a_1, -a_2, \dots, -a_N) \quad (\text{F.20})$$

Inner Products

The inner product of two vectors ($|\alpha\rangle$ and $|\beta\rangle$), is a complex number (which we write $\langle\alpha|\beta\rangle$), with the following properties:

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*, \text{ and } \langle\alpha|\alpha\rangle = \begin{cases} 0 & \text{if } |\alpha\rangle = |0\rangle \\ > 0 & \text{if } |\alpha\rangle \neq |0\rangle \end{cases}. \quad (\text{F.21})$$

And:

$$\langle \alpha | (b|\beta\rangle + c|\gamma\rangle) \rangle = b\langle \alpha | \beta \rangle + c\langle \alpha | \gamma \rangle. \quad (\text{F.22})$$

A vector space with an inner product is called an inner product space. Because Eq. (F.21), the inner product of any vector with itself is a nonnegative number, meaning its corresponding square root is real. We call this the norm of the vector:

$$\|\alpha\| = \sqrt{\langle \alpha | \alpha \rangle}. \quad (\text{F.23})$$

A vector is said to be normalized if its norm is equal to 1. Two vectors whose inner product equals zero are called orthogonal. A collection of mutually orthogonal normalized vectors is called an orthonormal set:

$$\langle \alpha_i | \alpha_j \rangle = \delta_{ij}. \quad (\text{F.24})$$

It is always possible and convenient to work using an orthonormal basis. If this is the case, then the inner product of two vectors can be written very neatly in terms of their components:

$$\langle \alpha | \beta \rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_N^* b_N. \quad (\text{F.25})$$

The norm becomes:

$$\|\alpha\| = \sqrt{|a_1|^2 + |a_2|^2 + \dots + |a_N|^2}. \quad (\text{F.26})$$

And the components of vector given by Eq. (F.15) are:

$$a_n = \langle \theta_n | \alpha \rangle. \quad (\text{F.27})$$

[Therefore, the wavefunctions given by Eq. (F.12) form an inner product space of dimension that is infinite. These functions form an orthonormal set because:

$$\langle \varphi_{n'} | \varphi_n \rangle = \int_0^L [\varphi_{n'}(x)]^* \varphi_n(x) dx = \frac{2}{L} \int_0^L \sin\left(\frac{n'\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = \delta_{n'n}. \quad (\text{F.28})$$

Any other vector, $|\phi\rangle$, can be represented as:

$$|\phi\rangle = \phi(x) = \sum_{n=1}^{+\infty} c_n \varphi_n(x), \quad \text{with } c_n = \langle \theta_n | \phi \rangle = \sum_{n=1}^{+\infty} c_n \langle \varphi_{n'} | \varphi_n \rangle. \quad (\text{F.29})$$

Linear Transformations

[In quantum mechanics, operators are linear transformations that transform any vector (wavefunction) of an inner product space into another vector (function) of the same space:

$$\widehat{T}|\alpha\rangle = |\gamma\rangle. \quad (\text{F.30})$$

The transformation is linear if and only if for any two vectors and scalars:

$$\widehat{T}(a|\alpha\rangle + b|\beta\rangle) = a\widehat{T}|\alpha\rangle + b\widehat{T}|\beta\rangle. \quad (\text{F.31})$$

In linear algebra, the theory of linear transformations is related to the theory of matrices.

[In this book, we have primarily used operators in their differential form (Table 2.1 in Chap. 2 contains some instances). This corresponds to the formulation of wave mechanics in quantum mechanics. Quantum mechanics can also be formulated in an equivalent matrixial form. Historically, Heisenberg's matrixial formulation of quantum mechanics was discovered shortly before the Schrödinger wave-based proposal. Below, we will summarize some important results of the theory of linear transformations in its matrixial form].

Let us suppose an inner product space of finite dimension (N), and an orthonormal set that allows for writing any vector (function) of the space as:

$$|\phi\rangle = \phi(x) = \sum_{j=1}^N d_j |\theta_j\rangle, \quad \text{with } d_j = \langle \theta_j | \phi \rangle, \quad \text{and } \langle \theta_{j'} | \theta_j \rangle = \delta_{j'j}. \quad (\text{F.32})$$

There is then a one-to-one correspondence between $\phi(x)$ and the list of N complex numbers (d_1, d_2, \dots, d_N) . We can then represent a linear transformation of the vector $|\phi\rangle$ in two equivalent forms:

$$\widehat{T}|\phi\rangle = |\gamma\rangle = \sum_{j=1}^N c_j |\theta_j\rangle. \quad (\text{F.33})$$

Or:

$$\widehat{T} \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_N \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_N \end{bmatrix}. \quad (\text{F.34})$$

The second form suggests that the linear transformation can be represented by a matrix. From Eq. (F.32), it follows that:

$$\langle \theta_i | \hat{T} | \phi \rangle = \sum_{j=1}^N d_j \langle \theta_i | \hat{T} | \theta_j \rangle. \quad (\text{F.35})$$

From Eq. (F.33), it follows that:

$$\langle \theta_i | \gamma \rangle = c_i. \quad (\text{F.36})$$

Therefore, from Eqs. (F.33) to (F.36), it follows that:

$$\langle \theta_i | \hat{T} | \phi \rangle = \sum_{j=1}^N d_j \langle \theta_i | \hat{T} | \theta_j \rangle = \langle \theta_i | \gamma \rangle = c_i. \quad (\text{F.37})$$

Or, in matrix notation:

$$\hat{T} \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_N \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_N \end{bmatrix}, \text{ with } \hat{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{21} & T_{22} & \dots & T_{2N} \\ \dots & \dots & \dots & \dots \\ T_{N1} & T_{N1} & \dots & T_{NN} \end{pmatrix}, \text{ and } T_{ij} = \langle \theta_i | \hat{T} | \theta_j \rangle. \quad (\text{F.38})$$

The study of linear transformations then reduces to the theory of matrices. The sum and multiplication of linear transformations are defined and have the properties of the sum and multiplication of matrices:

$$(\hat{S} + \hat{T})|\alpha\rangle = \hat{S}|\alpha\rangle + \hat{T}|\alpha\rangle \iff (\hat{S} + \hat{T})_{ij} = \hat{S}_{ij} + T_{ij}. \quad (\text{F.39})$$

The product of two linear transformations is the net effect of performing them in succession. The multiplication of matrices is not commutative, so the order in which the linear transformations are applied is important:

$$(\hat{S}\hat{T})|\alpha\rangle = \hat{S}(\hat{T}|\alpha\rangle) \neq (\hat{T}\hat{S})|\alpha\rangle = \hat{T}(\hat{S}|\alpha\rangle). \quad (\text{F.40})$$

The elements of the matrix resulting from the multiplication of two matrices are:

$$(\hat{S}\hat{T})_{ij} = \sum_{k=1}^N \hat{S}_{ik} T_{kj}. \quad (\text{F.41})$$

The commutator of two matrices is:

$$[\hat{S}, \hat{T}] = \hat{S}\hat{T} - \hat{T}\hat{S}. \quad (\text{F.42})$$

Two matrices commutative if and only if their commutator is null.

[Heisenberg's uncertainty principle is a consequence of the non-commutativity of the multiplication of matrices. In wave quantum mechanics, a quantum particle cannot simultaneously have well-determined values of its position and linear momentum because these operators do not commute:

$$\hat{x}\hat{p}\varphi(x) = -i\hbar x \frac{\partial}{\partial x}\varphi(x) \neq \hat{p}\hat{x}\varphi(x) = -i\hbar \frac{\partial}{\partial x}[x\varphi(x)]. \quad (\text{F.43})$$

The transpose of a matrix is the same set of elements, but with rows and columns interchanged:

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{21} & T_{22} & \dots & T_{2N} \\ \dots & \dots & \dots & \dots \\ T_{N1} & T_{N2} & \dots & T_{NN} \end{pmatrix} \Rightarrow \tilde{T} = \begin{pmatrix} T_{11} & T_{21} & \dots & T_{N1} \\ T_{12} & T_{22} & \dots & T_{N2} \\ \dots & \dots & \dots & \dots \\ T_{1N} & T_{2N} & \dots & T_{NN} \end{pmatrix}. \quad (\text{F.44})$$

Notice and note the hat ~ used for symbolizing the transpose matrix. Also, make a note that the transpose of a column matrix is a row matrix and vice versa:

$$|\phi\rangle \leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_N \end{bmatrix} \Rightarrow \tilde{|\phi\rangle} \leftrightarrow (\phi_1, \phi_2, \dots, \phi_N) \quad (\text{F.45})$$

The transpose of a product of matrices is the product of the transposes in reverse order:

$$\text{If } \hat{P} = \hat{S}\hat{T} \text{ then } \tilde{P} = \tilde{T}\tilde{S}. \quad (\text{F.46})$$

The complex transpose conjugate of a matrix (T^\dagger) is the complex conjugate of its transpose:

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{21} & T_{22} & \dots & T_{2N} \\ \dots & \dots & \dots & \dots \\ T_{N1} & T_{N2} & \dots & T_{NN} \end{pmatrix} \Rightarrow T^\dagger = (\tilde{T})^* = \begin{pmatrix} T_{11}^* & T_{21}^* & \dots & T_{N1}^* \\ T_{12}^* & T_{22}^* & \dots & T_{N2}^* \\ \dots & \dots & \dots & \dots \\ T_{1N}^* & T_{2N}^* & \dots & T_{NN}^* \end{pmatrix} \quad (\text{F.47})$$

T^\dagger is also known as the Hermitian conjugate or adjoint of the matrix T . A square matrix is Hermitian if it is equal to its Hermitian conjugate:

$$\hat{T} = T^\dagger \quad (\text{F.48})$$

For instance, the following matrix is Hermitian:

$$\hat{T} = \begin{pmatrix} 1 & i & i \\ -i & 1 & -i \\ -i & i & 1 \end{pmatrix}. \quad (\text{F.49})$$

The Hermitian conjugate of a product of matrices is the product of the adjoints in reverse order:

$$\text{If } \hat{P} = \hat{S}\hat{T} \text{ then } P^\dagger = T^\dagger S^\dagger \quad (\text{F.50})$$

Equation (F.48) gives a definition of a Hermitian transformation. An alternative definition of a Hermitian linear transformation involves the inner product of any two vectors $|\alpha\rangle$ and $|\beta\rangle$ in the following way:

$$T^\dagger \alpha | \beta = \alpha | \hat{T} \beta \quad (\text{F.51})$$

This means that a linear transformation is Hermitian if and only if the inner product of the vectors $T^\dagger |\alpha\rangle$ and $|\beta\rangle$ ($\langle T^\dagger \alpha | \beta \rangle$) is equal to inner product of the vectors:

$$|\alpha\rangle \text{ and } \hat{T} |\beta\rangle \left(\langle \alpha | \hat{T} \beta \rangle \right).$$

Most of the operators (linear transformations) used in quantum mechanics are Hermitian. For instance, in Chap. 2, the time independent Schrödinger equation for a particle in the one-dimensional infinite well is:

$$\hat{H} \varphi_n(x) = E_n \varphi_n(x). \quad (\text{F.52})$$

Therefore:

$$\langle \varphi_{n'} | \hat{H} \varphi_n(x) \rangle = E_n \langle \varphi_{n'} | \varphi_n \rangle = E_n \delta_{n'n}. \quad (\text{F.53})$$

Also:

$$\langle \hat{H} \varphi_n(x) | \varphi_{n'} \rangle = E_n \langle \varphi_n | \varphi_{n'} \rangle = E_n \delta_{n'n}. \quad (\text{F.54})$$

This implies that the total energy operator is Hermitian:

$$\hat{H} = \hat{H}^\dagger \quad (\text{F.55})$$

Eigenvectors and Eigenvalues

Eigenvector equations are equations of the form:

$$\hat{T} |\alpha\rangle = \lambda |\alpha\rangle. \quad (\text{F.56})$$

The vectors ($|\alpha\rangle$) and scalars (λ) satisfying Eq. (F.56) are name eigenvectors and eigenvalues, respectively.

[Time independent Schrödinger equations are eigenvector equations. A large part of most introductory courses of quantum mechanics is dedicated to the development of mathematical skills needed for solving eigenvector equations].

The eigenvectors and eigenvalues of Hermitian transformations (operators) have three important properties, which will be explained below.

First, the eigenvalues of a Hermitian transformation are real.

Demonstration

Let λ be an eigenvalue of Eq. (F.56), with $|\alpha\rangle \neq |0\rangle$. Therefore:

$$\langle\alpha|\hat{T}\alpha\rangle = \langle\alpha|\lambda\alpha\rangle = \lambda\langle\alpha|\alpha\rangle. \quad (\text{F.57})$$

Meanwhile, if the transformation is Hermitian, then:

$$\langle\alpha|\hat{T}\alpha\rangle = T^\dagger\langle\alpha|\alpha\rangle = \lambda^*\langle\alpha|\alpha\rangle \quad (\text{F.58})$$

But due to Eq. (F.21), $\langle\alpha|\alpha\rangle \neq 0$, so $\lambda = \lambda^*$, and hence λ is real.

[In quantum mechanics, as discussed Sect. 2.3, a Hermitian operator is associated to each observable physical magnitude. The reason for this is that measurements of physical magnitudes always return as real numbers. The real eigenvalues of the corresponding eigenvector equation are the only possible values of the physical magnitude associated to a Hermitian operator].

Second, the eigenvectors of a Hermitian transformation belonging to distinct eigenvalues are orthogonal.

Demonstration

Suppose that:

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle, \text{ and } \hat{T}|\beta\rangle = \mu|\beta\rangle, \text{ with } \lambda \neq \mu. \quad (\text{F.59})$$

Then:

$$\langle\alpha|\hat{T}|\beta\rangle = \langle\alpha|\mu\beta\rangle = \mu\langle\alpha|\beta\rangle. \quad (\text{F.60})$$

If the transformation is Hermitian:

$$\langle\alpha|\hat{T}|\beta\rangle = T^\dagger\langle\alpha|\beta\rangle = \lambda\langle\alpha|\beta\rangle = \lambda^*\langle\alpha|\beta\rangle \quad (\text{F.61})$$

But $\lambda = \lambda^*$ (from the first property), and $\lambda \neq \mu$, by assumption, so $\langle\alpha|\beta\rangle = 0$.

Third, the eigenvectors of a Hermitian transformation span the vector space. This property will be given here without demonstration.

[These properties together, with the linearity of the Schrödinger equation, justify the assertion made in Chap. 2.58 about Eq. (2.7) being the general solution of Eq. (2.49). This also justifies Eq. (2.59)].

Hilbert Space

A complete inner product space is called a Hilbert space. Complete could mean that the space is formed by all square-integrable functions on the interval $-1 < x < +1$. Technically, it is called $L_2(-1, +1)$. More generally, the set of all square-integrable functions in the interval $a < x < b$ is $L_2(a, b)$. The Hilbert space $L_2(-\infty, +\infty)$ is most often used in quantum mechanics.

[For instance, Eq. (F.28) shows that the wavefunction $\varphi_n(x)$ are vectors of the Hilbert space $L_2(-\infty, +\infty)$].

Annex G: About the Non-linearity of the PPGP Equations

The mathematical formalism of quantum mechanics (linear algebra, linear transformations, inner product spaces, and Hilbert spaces), requires linear wave equations. The Schrödinger equation is a linear equation. Also, the Klein-Gordon and Dirac equations are linear. However, the PPGP equations are not strictly linear.

For instance, let us consider the Schrödinger-like PPGP equation for a spin- $(s = 0)$ particle in a constant potential (Eq. 3.86):

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} \Psi + V\Psi, \quad \text{with } \mu = \left(1 + \frac{E - V}{2mc^2}\right)m. \quad (\text{G.1})$$

This equation is not linear because μ depends on E . If Ψ_1 and Ψ_2 are respectively distinct solutions of Eq. (G.1) with $E = E_1$ and $E = E_2$, respectively, then the effective mass of the relativistic quantum particle is different in these quantum states:

$$\text{If } E_1 \neq E_2 \Rightarrow \mu_1 = \left(1 + \frac{E_1 - V}{2mc^2}\right)m \neq \mu_2 = \left(1 + \frac{E_2 - V}{2mc^2}\right)m. \quad (\text{G.2})$$

Therefore, Ψ_1 satisfies the following equation:

$$i\hbar \frac{\partial}{\partial t} \Psi_1 = -\frac{\hbar^2}{2\mu_1} \frac{d^2}{dx^2} \Psi_1 + V\Psi_1. \quad (\text{G.3})$$

While Ψ_2 satisfies a different equation:

$$i\hbar \frac{\partial}{\partial t} \Psi_2 = -\frac{\hbar^2}{2\mu_2} \frac{d^2}{dx^2} \Psi_2 + V\Psi_2. \quad (\text{G.4})$$

However, in general, a linear combination of these two solutions:

$$\Psi = a\Psi_1 + b\Psi_2. \quad (\text{G.5})$$

It is a solution of neither Eq. (G.3) nor (G.4), and it is not a solution of Eq. (G.1). Nevertheless, as demonstrated in Annex A, we can find the solution of the Klein-Gordon equation with $E_T = E + mc^2$ by solving Eq. (G.1). This means that the wavefunction Ψ given by Eq. (G.5) is a solution of the Klein-Gordon equation.

Similarly, as discussed in Annex A, by solving the following complementary Schrödinger-like PPGP equation:

$$i\hbar \frac{\partial}{\partial t} \Omega = -\frac{\hbar^2}{2\mu'} \frac{d^2}{dx^2} \Omega + V\Omega, \quad \text{with } \mu' = \left(-1 + \frac{E' - V}{2mc^2}\right)m. \quad (\text{G.6})$$

We can find two distinct solutions Ω_1 and Ω_2 of Eq. (G.6) with $E' = E'_1$ and $E' = E'_2$, which are also solutions of the Klein-Gordon equation with $E_T = E' - mc^2$. The effective mass of the relativistic quantum particle is different in these quantum states:

$$\text{If } E'_1 \neq E'_2 \Rightarrow \mu'_1 = \left(-1 + \frac{E'_1 - V}{2mc^2}\right)m \neq \mu'_2 = \left(-1 + \frac{E'_2 - V}{2mc^2}\right)m. \quad (\text{G.7})$$

Therefore, Ω_1 satisfies the following equation:

$$i\hbar \frac{\partial}{\partial t} \Omega_1 = -\frac{\hbar^2}{2\mu'_1} \frac{d^2}{dx^2} \Omega_1 + V\Omega_1. \quad (\text{G.8})$$

While Ω_2 satisfies a different equation:

$$i\hbar \frac{\partial}{\partial t} \Omega_2 = -\frac{\hbar^2}{2\mu'_2} \frac{d^2}{dx^2} \Omega_2 + V\Omega_2. \quad (\text{G.9})$$

However, in general, a linear combination of these two solutions:

$$\Omega = a\Omega_1 + b\Omega_2. \quad (\text{G.10})$$

It is a solution of neither Eq. (G.8) nor (G.9), and it is not a solution of Eq. (G.6). Nevertheless, the wavefunction Ω , given by Eq. (G.10), is a solution of the Klein-Gordon equation. Consequently, a general solution of the Klein-Gordon equation can be written as a linear superposition of all the distinct solutions of Eqs. (G.1) and (G.6).

For similar reasons, the Pauli-like PPGP equation for a spin- $(s = 1/2)$ charged particle, in a constant electrostatic potential, is also non-linear:

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{\sigma} \cdot \hat{\mathbf{p}} \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \hat{\mathbf{p}} \Psi + V\Psi, \quad V = eA_o, \quad \text{with } \mu = \left(1 + \frac{E - V}{2mc^2}\right)m. \quad (\text{G.11})$$

In this case, μ is constant, so Eq. (G.11) can be simplified to:

$$i\hbar \frac{\partial}{\partial t} \Psi = \frac{(\hat{\sigma} \cdot \hat{\mathbf{p}})^2}{2\mu} \Psi + V\Psi = -\frac{\hbar^2}{2\mu} \nabla^2 \Psi + V\Psi. \quad (\text{G.12})$$

Therefore, each component of the spinor Ψ satisfies de Grave de Peralta (GP) equation:

$$i\hbar \frac{\partial}{\partial t} \psi_i = -\frac{\hbar^2}{2\mu} \nabla^2 \psi_i + V\psi_i, \text{ with } \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, i = 1, 2. \quad (\text{G.13})$$

In the non-relativistic limit, $E - V \ll mc^2$, the GP equation coincides with the Schrödinger equation. However, in general, the GP equation is nonlinear because μ depends on E . As discussed above, if ψ_1 and ψ_2 are distinct solutions of Eq. (G.13) with $E = E_1$ and $E = E_2$, respectively, then the effective mass of the relativistic quantum particle is different in these quantum states (Eq. G.2). Therefore, ψ_1 satisfies the following equation:

$$i\hbar \frac{\partial}{\partial t} \psi_1 = -\frac{\hbar^2}{2\mu_1} \nabla^2 \psi_1 + V\psi_1. \quad (\text{G.14})$$

While ψ_2 satisfies a different equation:

$$i\hbar \frac{\partial}{\partial t} \psi_2 = -\frac{\hbar^2}{2\mu_2} \frac{d^2}{dx^2} \psi_2 + V\psi_2. \quad (\text{G.15})$$

However, in general, a linear combination of these two solutions:

$$\psi = a\psi_1 + b\psi_2. \quad (\text{G.16})$$

It is a solution of neither Eq. (G.14) nor (G.15), and it is also not a solution of Eq. (G.13). Consequently, if the spinors Ψ_1 and Ψ_2 are distinct solutions of Eq. (G.11) with $E = E_1$ and $E = E_2$, respectively, then:

$$\Psi = a\Psi_1 + b\Psi_2. \quad (\text{G.17})$$

In general, it is not a solution of Eq. (G.11). Nevertheless, we can utilize Eq. (A.14) for constructing the Dirac wavefunction (Ψ_{Dl}) given by Eq. (A.11) that corresponds to the total energy of the particle $E_T = E_l + mc^2$. If the first spinor forming the bi-spinor Ψ_{Dl} is the following solution of Eq. (G.11):

$$\Psi_1 = \varphi_1 e^{-\frac{i}{\hbar} E_1 t}. \quad (\text{G.18})$$

Then, the second spinor is:

$$\Omega_1 = \chi_1 e^{-\frac{i}{\hbar} E_1 t}. \quad (\text{G.19})$$

In Eq. (G.19), the spinor χ_1 is related to the spinor φ_1 through Eq. (A.14). Therefore:

$$\Psi_{D1} = \begin{pmatrix} \Psi_1 \\ \Omega_1 \end{pmatrix}, \text{ corresponds to } E_T = E'_1 - mc^2 \quad (\text{G.20})$$

Following the same approach, we can obtain that the Dirac wavefunction that corresponds to Ψ_2 is:

$$\Psi_{D2} = \begin{pmatrix} \Psi_2 \\ \Omega_2 \end{pmatrix}, \text{ corresponds to } E_T = E'_2 - mc^2 \quad (\text{G.21})$$

Consequently, a linear superposition of these two Dirac wavefunctions is also a solution of the Dirac equation:

$$\Psi_D = a\Psi_{D1} + b\Psi_{D2} \quad (\text{G.22})$$

Similarly, as discussed in Annex A, we could solve the following complementary Pauli-like PPGP equation:

$$i\hbar \frac{\partial}{\partial t} \Omega = \hat{\sigma} \cdot \hat{\mathbf{p}} \left[\frac{1}{2\mu'} \right] \hat{\sigma} \cdot \hat{\mathbf{p}} \Omega + V\Omega, \text{ with } \mu' = \left(-1 + \frac{E' - V}{2mc^2} \right) m. \quad (\text{G.23})$$

In this case μ' is constant, so Eq. (G.23) can be simplified to:

$$i\hbar \frac{\partial}{\partial t} \Omega = \frac{(\hat{\sigma} \cdot \hat{\mathbf{p}})^2}{2\mu'} \Omega + V\Omega = -\frac{\hbar^2}{2\mu} \nabla^2 \Omega + V\Omega. \quad (\text{G.24})$$

Therefore, each component of the spinor Ω satisfies de complementary GP equation:

$$i\hbar \frac{\partial}{\partial t} \phi_i = -\frac{\hbar^2}{2\mu'} \nabla^2 \phi_i + V\phi_i, \text{ with } \Omega = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, i = 1, 2. \quad (\text{G.25})$$

Note that in the non-relativistic limit, $E' - V \ll mc^2$, the complementary GP equation formally coincides with the Schrödinger equation corresponding to a particle with negative mass. Like the GP equation, the complementary GP equation is nonlinear because μ' depends on E' . Again, if ϕ_1 and ϕ_2 are distinct solutions of Eq. (G.25), with $E' = E'_1$ and $E' = E'_2$, then the effective mass of the relativistic quantum particle is different in these certain quantum states (Eq. G.7). Therefore, ϕ_1 satisfies the following equation:

$$i\hbar \frac{\partial}{\partial t} \phi_1 = -\frac{\hbar^2}{2\mu'_1} \nabla^2 \phi_1 + V \phi_1. \quad (\text{G.26})$$

While ϕ_2 satisfies a different equation:

$$i\hbar \frac{\partial}{\partial t} \phi_2 = -\frac{\hbar^2}{2\mu'_2} \frac{d^2}{dx^2} \phi_2 + V \phi_2. \quad (\text{G.27})$$

However, in general, a linear combination of these two solutions:

$$\phi = a\phi_1 + b\phi_2. \quad (\text{G.28})$$

It is a solution of neither Eq. (G.26) nor (G.27), and it is also not a solution of Eq. (G.24). Consequently, if the spinors Ω_1 and Ω_2 are distinct solutions of Eq. (G.24), with $E' = E'_1$ and $E' = E'_2$, then:

$$\Omega = a\Omega_1 + b\Omega_2. \quad (\text{G.29})$$

In general, it is not a solution of Eq. (G.24). Nevertheless, we can utilize Eq. (A.17) for constructing the Dirac wavefunction ($\Psi_{D'1}$) given by Eq. (A.11) that corresponds to the total energy of the particle $E_T = E'_1 - mc^2$. If the second spinor forming the bi-spinor Ψ_{D1} is the following solution of Eq. (G.24):

$$\Omega_1 = \chi_1 e^{-\frac{i}{\hbar} E'_1 t}. \quad (\text{G.30})$$

Then, the first spinor is:

$$\Psi_1 = \varphi_1 e^{-\frac{i}{\hbar} E'_1 t}. \quad (\text{G.31})$$

In Eq. (G.31), the spinor φ_1 is related to the spinor χ_1 through Eq. (A.17). Therefore:

$$\Psi_{D'1} = \begin{pmatrix} \Psi_1 \\ \Omega_1 \end{pmatrix}, \text{ corresponds to } E_T = E'_1 - mc^2 \quad (\text{G.32})$$

Following the same approach, we can obtain that the Dirac wavefunction that corresponds to Ω_2 is:

$$\Psi_{D'2} = \begin{pmatrix} \Psi_2 \\ \Omega_2 \end{pmatrix}, \text{ corresponds to } E_T = E'_2 - mc^2 \quad (\text{G.33})$$

Consequently, a linear superposition of these two Dirac wavefunctions is also a solution of the Dirac equation:

$$\Psi_{D'} = a\Psi_{D'1} + b\Psi_{D'2} \quad (\text{G.34})$$

Finally, a general solution of the Dirac equation can be written as a linear superposition of all the distinct Dirac functions Ψ_D and $\Psi_{D'}$ given by Eqs. (G.22) and (G.34), respectively.

Annex H: The Pedagogical Value of the PPGP Equations

Quantum mechanics is a century old discipline. Numerous amounts of wonderful books have been written about non-relativistic and relativistic quantum mechanics. Undergraduate physics, chemistry, and engineering students often receive at least an introductory quantum mechanics course. Commonly, introductory quantum mechanics courses present a comprehensive approach to non-relativistic quantum mechanics. However, relativistic quantum mechanics is often not included in these introductory courses. Currently, only some graduate students majoring in Physics and Mathematics receive a relativistic quantum mechanics course. This is because additional mathematical skills are required for mastering relativistic quantum mechanics. As a result, a relatively large group of professionals are familiar with non-relativistic quantum mechanics, but only a small elite of theoretical physicists have a decent understanding of relativistic quantum mechanics.

This situation represents an undesirable barrier that slows the broad dissemination of one of the best scientific theories ever developed. For unleashing the full potential of relativistic quantum mechanics, it would be desirable to increase the number of professionals capable of understanding the fundamental ideas involved in the intersection of the two greatest physical theories of the Twentieth Century: relativity and quantum mechanics.

In most introductory quantum mechanics courses, an immense amount of time is dedicated to developing the mathematical skills required to solve the Schrödinger equation corresponding to different physical situations. There is fortunate formal similitude between the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \quad (\text{H.1})$$

And the Schrödinger-like PPGP equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2\mu}\nabla^2\Psi + V\Psi, \text{ with } \mu = \left[1 + \frac{E - V}{2mc^2}\right]m. \quad (\text{H.2})$$

This formal similitude provides a huge pedagogical opportunity for simultaneously introducing the students to non-relativistic and relativistic quantum mechanics. This is because the same mathematical skills are needed for solving Eqs. (H.1) and (H.2). Moreover, the instructor could gradually introduce the students to the apparent

simplicity of Eq. (H.2). Like m , the effective mass of the relativistic quantum particle (μ) is constant in problems where V is constant (Sects. 3.6, 3.7, and 5.1). In this case, Eq. (H.2) reduces to the Grave de Peralta (GP) equation:

$$i\hbar\frac{\partial}{\partial t}\psi_{GP} = -\frac{\hbar^2}{2\mu}\nabla^2\psi_{GP} + V\psi_{GP}, \text{ with } \mu = \frac{1+\gamma}{2}m = \text{constant.} \quad (\text{H.3})$$

In Eq. (H.3), γ is the Lorentz factor of special theory of relativity. This provides a direct link between non-relativistic quantum mechanics ($\gamma = 1$) and the special theory of relativity. The richness of Eq. (H.2) can be easily explored in physical situations where V is not constant. The relativistic harmonic oscillator (Sect. 4.1) and the “discovery” of the Klein Paradox (Sect. 4.2) are two excellent pedagogical examples of what brings special relativity to quantum mechanics.

The motif (explicit or not) of a typical quantum mechanics introductory course could be condensed in the following phrase: there is a wave associated to any quantum particle and the wavefunction associated to this wave can be obtained by solving the Schrödinger equation. However, in relativistic quantum mechanics the wave equation that should be solved (for spin- $(s = 0)$ particles) is the Klein-Gordon equation. Lorentz invariant (covariant) wave equations bring the existence of “exotic” states in relativistic quantum mechanics (see Chap. 3). We are extremely fortunate because of the existence of the complementary Schrödinger-like PPGP equation:

$$i\hbar\frac{\partial}{\partial t}\Omega = -\frac{\hbar^2}{2\mu'}\nabla^2\Omega + V\Omega, \text{ with } \mu' = \left[-1 + \frac{E' - V}{2mc^2}\right]m. \quad (\text{H.4})$$

Also, this is formally equal to the Schrödinger equation. Therefore, new mathematical skills are not necessary to solve it. By solving Eq. (H.4), we can obtain the wavefunctions of the “exotic” states that special relativity brings to quantum mechanics. Moreover, Eq. (H.4) is related through Theorem III (Annex B) with the antiparticle’s Schrödinger-like PPGP equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = -\frac{\hbar^2}{2\mu_a}\nabla^2\Psi_a - V\Psi_a, \text{ with } \mu_a = \left[1 + \frac{E_a + V}{2mc^2}\right]m. \quad (\text{H.5})$$

This means that the existence of antiparticles, which is a theoretical prediction of relativistic quantum mechanics, could be explained by exclusively utilizing Schrödinger-like equations. This also allows the instructor to naturally introduce the students to the Dirac’s Sea model and the Hole Theory (Sects. 3.3 and 4.3).

In addition to learning how to solve the Schrödinger equation in common introductory courses of quantum mechanics, students typically learn about the existence of particles with spin. This is because atoms are formed by electrons, which are particles with spin $s = 1/2$. Students learn that it is not the Schrödinger equation, but the Pauli equation that should be solved for describing the interaction of a beam of electrons with an external electromagnetic field (Eq. 6.21):

$$i\hbar \frac{\partial}{\partial t} \Psi = \frac{[\hat{\sigma} \cdot (\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A})]^2}{2m} \Psi + V \Psi, \text{ with } V = eA_o, \text{ and } \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (\text{H.6})$$

Amazingly, we are pedagogically fortunate again because there exist a Pauli-like PPGP equation that in the nonrelativistic limit coincides with the Pauli equation (Eq. A.9):

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \Psi + V \Psi. \quad (\text{H.7})$$

Due to Theorem II in Annex A, for the Hydrogen atom, the energy of the stationary states corresponding to Eq. (H.7) are exactly the energies predicted by the Dirac equation (Sect. 6.5):

$$E_{n,j} = \frac{mc^2}{\sqrt{1 + \frac{Z^2 \alpha^2}{\left[n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - Z^2 \alpha^2} \right]^2}}} - mc^2, \text{ with } j = \frac{1}{2}, \frac{3}{2}, \dots, n - 1/2. \quad (\text{H.8})$$

A direct comparison of Eqs. (H.7) and (H.6) give the instructor a rare pedagogical opportunity. This is to pinpoint exactly why the Pauli equation fails us if we try to use it for describing the electron in the Hydrogen atom. The reason is that only if μ is constant, then:

$$\hat{\sigma} \cdot \hat{\mathbf{p}} \left[\frac{1}{2\mu} \right] \hat{\sigma} \cdot \hat{\mathbf{p}} = \frac{[\hat{\sigma} \cdot \hat{\mathbf{p}}]^2}{2\mu} = -\frac{\hbar^2}{2\mu} \nabla^2. \quad (\text{H.9})$$

The series of fortunate pedagogical events does not finish here. Also, the “exotic” spinors Ω that correspond to the solutions of the Dirac equation (Theorem II, Annex A) can be obtained by solving the complementary Pauli-like PPGP equation (Eq. A.10):

$$i\hbar \frac{\partial}{\partial t} \Omega = \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \left[\frac{1}{2\mu'} \right] \hat{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right) \Omega + V \Omega. \quad (\text{H.10})$$

Amazingly, the values of μ and μ' do not depend on the nature (spin value) of the particle:

$$\mu = \left[1 + \frac{E - V}{2mc^2} \right] m \text{ and } \mu' = \left[-1 + \frac{E' - V}{2mc^2} \right] m. \quad (\text{H.11})$$

The reason for this pedagogical simplicity is that, as shown in Sect. 3.1, the equation given the value of μ can be obtained without referring to quantum mechanics. From the simple relativity equations:

$$E_T = K + V + mc^2 = E + mc^2, \text{ and } K = (\gamma - 1)mc^2. \quad (\text{H.12})$$

It follows the following useful formula for the Lorentz factor (γ):

$$\gamma = 1 + \frac{E - V}{mc^2}. \quad (\text{H.13})$$

Defining μ as:

$$\mu = \frac{(\gamma + 1)}{2}m \Rightarrow \gamma = \frac{2\mu}{m} - 1. \quad (\text{H.14})$$

From Eqs. (H.13) and (H.14), it follows the given formula for μ given by Eq. (H.11). To obtain the formula for μ' given by Eq. (H.11), in addition to relativity, we only need the Hole Theory, which is a pure relativistic quantum mechanics model that is valid for any relativistic quantum particle. The antiparticle has the same mass but opposite electrical charge than the particle. However, the antiparticle is just another particle. Therefore, if the antiparticle interacts with the same external world than the corresponding particle, then Eq. (H.13) should be modified in the following way for the antiparticle:

$$\gamma_a = 1 + \frac{E_a + V}{mc^2}. \quad (\text{H.15})$$

The Hole Theory states that $E_a = -E'$. A consequence of this is that $\gamma_a = -\gamma'$ (Eq. 3.34). This is valid for any kind of particle. Therefore:

$$-\gamma' = 1 + \frac{-E' + V}{mc^2} \Rightarrow \gamma' = -1 + \frac{E' - V}{mc^2}. \quad (\text{H.16})$$

Defining μ' as:

$$\mu' = \frac{(\gamma' - 1)}{2}m \Rightarrow \gamma' = \frac{2\mu'}{m} + 1. \quad (\text{H.17})$$

From Eqs. (G.16) and (G.17), it follows the formula for μ' given by Eq. (H.11).

Summarizing the discussion above:

1. In traditional introductory quantum mechanics courses, students learn how to solve the Schrödinger and Pauli equations.
2. We present a pedagogical approach for simultaneously teaching nonrelativistic and relativistic quantum mechanics in this book. The approach presented only requires solving Schrödinger-like and Pauli-like equations. The approach presented is then compatible with the approach often used for teaching nonrelativistic quantum mechanics.

3. Formally, there is a fundamental difference between the Schrödinger and Pauli equations and the corresponding Schrödinger-like and Pauli-like PPGP equations. This is because m is a constant in the Schrödinger and Pauli equations, but μ and μ' are local functions on the position of the particle. This difference captures most of the consequences that special theory of relativity brings to quantum mechanics. The simplicity of the differences provides a pedagogical opportunity for simultaneously teaching nonrelativistic and relativistic quantum mechanics.

This book is an example of how we could simultaneously teach and learn nonrelativistic and relativistic quantum mechanics in an introductory quantum mechanics course. The authors of this book wish and expect that future books based on similar approaches could be even better.

Annex I: The Heuristic Value of the PPGP Equations

There is latent tension between Physics and Mathematics. While mathematics is the language of physics, physics is not necessarily mathematics. There is some verity in the phrase “if you cannot explain a physical idea in Layman Terms, you still do not understand it”. An inadequate amount of mathematics may result in an imprecise physical theory, but an excessive amount of mathematics may result in a physical theory that is unintelligible for many.

Nonrelativistic quantum mechanics may be a physical theory with too much mathematics for a multitude of people. However, a century’s worth of teaching experience resulted in clever ways of disseminating the fundamental physical ideas included in it.

- What is quantum mechanics?—An amateur could ask

Today, most physicists, chemists, and many engineers could answer in a comprehensible way to the amateur:

- There are not any waves associated with a classical particle, but there is a wave associated with every quantum particle. Quantum mechanics is the physical theory that indicates how to calculate the wavefunction that describes how that wave behaves. Wavefunctions are solutions of wave equations. Specific mathematical skills are needed for solving wave equations. If you are interested, there are many pedagogical quantum mechanics books that are available to learn from.
- What is this thing entitling relativistic quantum mechanics? A nonprofessional could also ask.
- Well... you probably could not handle my answer! An arrogant and discourteous expert in quantum electrodynamics may say.

However, a reader of this book may answer something like this:

- As I told you before, there are no waves associated with a classical particle. However, I should apologize to you for my previous response.
- I should have said that, in the nonrelativistic quantum mechanics theory, there is a wave associated with a quantum particle. However, nonrelativistic quantum mechanics is only an approximated theory. This is because it is only valid if the energy of the particle is very small.
- Relativistic quantum mechanics is the precise theory where it is valid for any particle energy. To be precise, there are two waves associated with a single quantum particle. Relativistic quantum mechanics is the physical theory that indicates how to calculate the wavefunctions that describes how both waves behave. Wavefunctions are solutions of wave equations. A few mathematical skills are necessary for solving wave equations. If you are interested, I know of a pedagogical book that allows you to simultaneously learn relativistic and non-relativistic quantum mechanics.

This hypothetical dialog illustrates the heuristic value of the pedagogical approach in which this book is based. A simpler, but still precise, theoretical approach may facilitate thinking about the physical meaning of the mathematics involved in it. Moreover, it may facilitate thinking about possible consequences of the theory.

Excessive care about mathematical subtleties may impede non-specialists from adventuring in advanced mathematical physical theories. For instance, Dirac, a mathematical genius, did not vacillate successfully using the so-called “Delta of Dirac function”. We know today that it is not strictly a function, but instead another mathematical object. The renormalization procedure in modern quantum field theories is another example of this kind.

In Chap. 8, several heuristic discussions are presented. They explore some hypothetical but far-reaching consequences due to the introduction of a special theory of relativity in quantum mechanics.

The simplest model of a relativistic electron in a heavy Hydrogen-like atom is a ($s = 0$)-particle trapped in a one-dimensional infinite well (Sect. 3.6). Clearly, this is not an extremely precise model. The spin of the electron is not 0 but 1/2. The interaction between the electron and the nucleus of the Hydrogen-like atom is Coulombic. Nevertheless, this simple model captures some fundamental properties of the electron in the Hydrogen-like atom: the atom is stable because the electron has a minimum possible energy value different than zero, and there is a discrete set of possible stationary quantum states.

As discussed in Sect. 3.6, this simple model implies that we should solve the following Schrödinger-like PPGP equation (Eq. H.2):

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2\mu} \nabla^2 \Psi + V \Psi, \text{ with } \mu = \left[1 + \frac{E - V}{2mc^2} \right] m. \quad (\text{I.1})$$

Although the simplicity of this equation is misleading, there are explanations for it. As discussed in Sects. 4.2 and 4.3, μ is not constant and it is negative outside the well. This means that the exact mathematical model includes solving the corresponding

complementary Schrödinger-like PPGP equation and considering the Klein paradox. However, we are interested in a simple theory that captures the essence of this simple model. With this goal in sight, it was supposed in Sect. (3.6) that $\Psi = 0$ outside of the well. This implies simplifying the equation to be solved from Eq. (I.1) to solving inside of the well the GP equation for a free particle:

$$i\hbar\frac{\partial}{\partial t}\psi_{GP} = -\frac{\hbar^2}{(\gamma+1)m}\nabla^2\psi_{GP}, \text{ with } \gamma = \left[1 + \frac{E}{mc^2}\right]. \quad (\text{I.2})$$

The energies of the stationary states of Eq. (I.2) are (Eq. 3.81):

$$E_n^{(rel)} = \frac{\hbar^2\pi^2n^2}{(1+\gamma)mL^2}, \text{ with } \gamma = \sqrt{1 + \frac{n^2}{4}\left(\frac{\lambda_C}{L}\right)^2}, \text{ and } \lambda_C = \frac{h}{mc}. \quad (\text{I.3})$$

In the non-relativistic limit ($\gamma = 1$), Eq. (I.3) reduces to:

$$E_n^{(rel)} = \frac{\hbar^2\pi^2n^2}{2mL^2}. \quad (\text{I.4})$$

The energy of the particle inside of the well is purely kinetic ($V = 0$). Therefore, we obtain using this simple model that the kinetic energy of an electron in the ground state of a Hydrogen-like atom is approximately:

$$K_{n=1}^{(rel)} \approx \frac{\hbar^2}{(1+\gamma)mr^2}, \text{ with } \gamma = \sqrt{1 + \frac{n^2}{4}\left(\frac{\lambda_C}{L}\right)^2}, \text{ and } \lambda_C = \frac{h}{mc}. \quad (\text{I.5})$$

We could use the nonrelativistic limit of Eq. (I.5) for the Hydrogen atom:

$$K_{n=1} \approx \frac{\hbar^2}{2mr^2}. \quad (\text{I.6})$$

These results are simple enough to use them in some heuristic discussions. For instance, in Sect. 8.1, Eq. (I.6) was used for making a crude estimate of the size of the Hydrogen atom. The basic idea is that the total energy of the electron in the Hydrogen atom should be equal to the sum of its nonrelativistic kinetic and potential energy (Eq. 8.1):

$$E(r) \approx \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\epsilon_0 r}. \quad (\text{I.7})$$

We should note that Eq. (I.7) is a heuristically built function. It is not a wave equation. Nevertheless, it allows us to grasp how big a Hydrogen atom is. The heuristic argument is that the function $E(r)$ should have a local extremum when r equals the radius of the Hydrogen atom. Interestingly, this value turns to be the Bohr radius

(Eq. 8.2):

$$r = r_B = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{1}{\alpha}\lambda_C, \text{ with } \lambda_C = \frac{\hbar}{m_e c} \quad (\text{I.8})$$

This coincidence suggests that our heuristic considerations can be trusted even if they are not exact. The heuristic approach is highly advantageous from the mathematical point of view. This is because the mathematical skills required for finding local extrema of a function are of lower level, if compared to the skills required for solving an equation with partial derivatives (a PPGP equation). In Sect. 7.2, Eq. (7.24) is successfully utilized for predicting the Chandrasekhar mass limit from Heuristic considerations. This further supports our heuristic considerations. We could then try to predict the size of heavy Hydrogen-like atoms in this way. We should start by substituting Eq. (I.7) by Eq. (8.3):

$$E(r) \approx \frac{\hbar^2}{(\gamma + 1)m_e r^2} - \frac{Ze^2}{4\pi\epsilon_0 r}. \quad (\text{I.9})$$

The heuristic argument is now that the function $E(r)$ should have a local extremum when r equals the radius of the heavy Hydrogen-like atom. We easily find that $E(r)$ has a local minimum when:

$$r = r_Z = a\sqrt{1 - \left(\frac{\lambda_C}{a}\right)^2}, \text{ with } a = \frac{r_B}{Z} \quad (\text{I.10})$$

As shown in Fig. 8.1, the size of the Hydrogen-like atom becomes undefined when $Z > 1/\alpha \approx 137$. This could be interpreted as a prediction about the impossibility of the natural existence of elements with $Z > 137$. Interestingly, an element with $Z > 118$ has never been discovered. All the heuristic predictions above are confirmed by experimental observations. Moreover, they make a lot of sense from the theoretical point of view.

We then reach a point where we could adventure ourselves to explore some possible far-reaching consequences of relativistic quantum mechanics. We should look for ideas that are controversial due to the mathematical complexity of “traditional” relativistic quantum mechanics. We may hope that the simplicity of the PPGP equations could bring some light to current controversial topics.

One of these controversial topics is the possible existence of a clear frontier between the quantum world and the classical world. Currently, many quantum mechanics experts believe that everything from a quark to the whole universe is quantum. Classical mechanics would then only be an approximate theory. The exact theory is relativistic quantum mechanics. Other physicists (experts included) argue that Newtonian mechanics can be successfully used for designing mechanical machines. This is because the macroscopic objects that surround us are as classical as they seem to be. For them, Schrödinger’s cat does not exist because cats are classical

living beings. Nobody has ever seen a cat that is simultaneously dead and alive. The general population would most likely agree with this opinion.

Interestingly, based on the relative simplicity of the PPGP equations, we could possibly give a heuristic answer to the question about the existence of a clear frontier between the quantum and the classical world. If possible, we should not wish to contradict known results of relativistic quantum mechanics since it is a very successful theory. Moreover, quantum electrodynamics was the first modern quantum field theory ever developed. Other quantum field theories forming the current Standard Model of particle physics were developed following the theory of quantum electrodynamics. Therefore, it is very difficult to contradict the more precise physical theory ever developed, as the experts in the field commonly say. If possible, we should consider looking for something external to “traditional” relativistic quantum mechanics.

We notice that an attractive interaction (proportional to $1/r^2$) between a quantum particle and the external world tends to spatially localize the wavefunction (scalar or spinor) Ψ . For instance, this happens in Hydrogen-like atoms. As discussed above, if this interaction is too strong, it can produce the collapse of Ψ to a point (Eq. I.10). Neither gravity nor a possible interaction of a quantum particle with itself are included in relativistic quantum mechanics. Therefore, a combination of these two actors may result in the existence of a clear frontier between the quantum and the classical world.

It is a straightforward extension of previous heuristic arguments to guess that a quantum particle could interact gravitationally with itself. As discussed in Sect. 8.2, if this could occur, then we could approximately capture the consequences of this by modifying Eq. (I.9) in the following way:

$$E(r) \approx \frac{\hbar^2}{2\mu r^2} - \frac{Gm^2}{r}, \quad \text{with } \mu = \frac{(\gamma + 1)}{2}m > 0. \quad (\text{I.11})$$

Now, the heuristic argument would be that the function $E(r)$ may contain a local extremum. This would correspond to a value of $r = r_m$, which refers to the “size” of the wavefunction Ψ that is associated to a free quantum particle. The charge of the particle is not present in Eq. (I.10). The antiparticle is just another particle with the same mass but contains the opposite electric charge of the particle. Therefore, Eq. (I.10) should also be valid for antiparticles. As shown in Fig. 8.2, $r_m > 0$ for any finite value of m in the nonrelativistic limit ($\gamma = 1$). However, $r_m \rightarrow 0$ if $m = m_P$ (Planck mass), when the effects produced by the introduction of special theory of relativity in quantum mechanics are considered ($\gamma \gg 1$). We could interpret the collapse of Ψ to a point as a transition from quantum to classical. The “dressed” quantum particle get “naked” when $m > m_P \approx 22 \mu\text{g}$. “Naked” particles with no wave associated to it should be classical particles. This is a surprise since it looks like an extended version of relativistic quantum mechanics agrees with the amateur opinion about the Schrödinger cat.

Moreover, we also notice that a repulsive interaction (proportional to $1/r^2$) between a quantum particle and the external world tends to spatially localize the “exotic” wavefunction (scalar or spinor) Ω . For instance, as discussed in Sect. 8.3, this happens

when a positron interacts with the nucleus of a Hydrogen-like atom (Fig. 8.5a). A possible Coulombic interaction of a particle, or antiparticle, with itself is not included in relativistic quantum mechanics. We could then wonder what the consequences would be if we consider an extended version of relativistic quantum mechanics where a charged particle or antiparticle could electrically repel itself.

Let us start considering antiparticles. Could this extended relativistic quantum mechanics theory explain a big mystery? Nonprofessionals do not care for antimatter since we are surrounded by matter. However, physicists care about the existence of antimatter. The existence of antimatter is a consequence of including relativity in quantum mechanics. Moreover, the Standard Model of particle physics predicts that we should be equally surrounded by matter and antimatter, but we are not surrounded by antimatter.

If an antiparticle with electric charge q could experience an internal Coulomb repulsion, as discussed in Sect. 8.3, we could heuristically argue that we could approximately capture the consequences of this assumption by modifying Eq. (I.11) in the following way:

$$E'_a(r) \approx \frac{\hbar^2}{2\mu'_a r^2} + \frac{Zq^2}{4\pi\epsilon_0 r} = -\frac{\hbar^2}{2\mu r^2} + \frac{Zq^2}{4\pi\epsilon_0 r}. \quad (\text{I.12})$$

As discussed in Annex E (Fig. E.2), Eq. (I.12) assumes that, if the self-interacting antiparticle is in the “exotic” quantum state that corresponds to Ω_a , then its energy is E'_a , its kinetical energy is negative, and its internal potential energy is positive. Now, the heuristic argument would be that the function $E'_a(r)$ may have a local extremum. This would correspond to a value of $r = r_q$, which refers to the “size” of the wavefunction Ω_a that is associated to a free quantum antiparticle. As shown in Fig. 8.7, $r_q \rightarrow 0$ if $|q| = q_P$ (Planck charge). We could interpret the collapse of Ω_a to a point as a transition from quantum to classical. The “dressed” quantum antiparticle get “naked” when $|q| > q_P \approx 11 e$. “Naked” antiparticles with no waves associated to them should be classical antiparticles.

This is a surprising result because from it, we could explain why antimatter does not surround us. Relativistic quantum mechanics explains the stability of antimatter atoms by the assumption of two waves associated with the cloud of Z positrons forming it. However, the extended version of quantum mechanics that we are considering, where a charged antiparticle repels itself, predicts that a cloud of more than 11 positrons should be classical. This is because Ω_a collapses to a point in this case. Therefore, no antimatter atoms with $Z > 11$ should exist. This heuristic prediction coincides with the physical reality that surrounds us.

However, we are surrounded by matter. This implies that Eq. (I.12) should not be valid for matter particles. We are heuristically constructing an extended version of relativistic quantum mechanics. This extended theory should explain the world as we see it. Therefore, in this extended theory, a charged particle should interact with itself in a different way than a charged antiparticle does. For instance, we could propose that for particles, Eq. (I.12) should be replaced by:

$$E'(r) \approx -\frac{\hbar^2}{2\mu r^2} - \frac{Ze^2}{4\pi\epsilon_0 r}. \quad (\text{I.13})$$

Clearly, $E'(r)$ does not have a local extremum. Consequently, the particle in “exotic” quantum states, associated to Ω , never collapses to a point. This means that free quantum particles can have any charge. This is why we are surrounded by matter but not by antimatter.

In summary, the relative simplicity of the PPGP equations facilitates the use of some interesting heuristic semiquantitative arguments. The heuristic discussions above allowed the filling of some notable holes in the existing quantum mechanics theory. Nevertheless, as we stated at the beginning of this Annex, there is latent tension between Physics and Mathematics. While mathematics is the language of physics, physics is not mathematics. An immense amount of mathematics may result in a physical theory that is unintelligible for many. However, an inadequate amount of mathematics may result in an imprecise physical theory. This means that the heuristic finding presented above should be received with both justified enthusiasm and skepticism. This is how science progresses from enthusiastic skepticism to careful proof or disproof of unique hypothetical theories.