

Hamilton-Jacobi and Lagrange Formulations of Relativistic Quantum Mechanics Wave Equations with Solutions with Only-Positive and Only-Negative Kinetic Energies

Luis Grave de Peralta^{1,2*}, Arquímedes Ruiz-Columbie³

¹Department of Physics and Astronomy, Texas Tech University, Lubbock, TX, USA

²Nano Tech Center, Texas Tech University, Lubbock, TX, USA

³Wind Energy Program, Texas Tech University, Lubbock, TX, USA

Email: *luis.grave-de-peralta@ttu.edu

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Abstract

Using the Hamilton-Jacobi and the Lagrange formalisms, a pair of relativistic quantum mechanics equations are obtained by abduction. These equations, in contrast with the Klein-Gordon and other relativistic quantum mechanics equations, have no solutions with both positive and negative kinetic energies. The equation with solutions with only positive kinetic energy values describes a spin-0 particle of mass m , which is moving at relativistic speeds in a scalar potential. The wavefunctions and the energies corresponding to the associated antiparticle can be obtained by solving the other equation, which only has solutions with negative kinetic energy values.

Keywords

Quantum Mechanics, Relativistic Quantum Mechanics

1. Introduction

The best-known relativistic quantum mechanics equations are the Klein-Gordon and Dirac equations [1] [2]. Both equations have solutions corresponding to a particle with mass m with both positive and negative kinetic energy values [1] [2]. Classical particles only can have positive kinetic energy values. The same occurs for the Schrödinger equation, which is the best-known non-relativistic quantum mechanics wave equation [3] [4]. Therefore, the existence of solutions of relativistic wave equations, that seems to correspond to a particle with nega-

tive kinetic energy, was a puzzle that conducted to the theoretical prediction of the existence of antiparticles [1] [2] [3] [4]. The existence of antiparticles is today a well-known experimental fact [1] [2] [3] [4]. Today, it is well-known that antiparticles also have positive kinetic energy values [1] [2] [3] [4]; nevertheless, the concept of antiparticle remains associated to the solutions of the Klein-Gordon and Dirac equations corresponding to negative values of the kinetic energy [1] [2].

Recently, an interesting relativistic quantum mechanics wave equation that does not have solutions corresponding to a particle with negative kinetic energy, has attracted our attention [5] [6] [7]. The authors of this work have been studying for a while the so-called general Grave de Peralta (gGP) equation [5] [6] [7] [8] [9]:

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2\mu} \nabla^2 \Psi + V\Psi, \text{ with } \mu = \frac{1+\gamma}{2}m. \quad (1)$$

This is a Schrödinger-like, but relativistic quantum mechanics equation, that describes the quantum states of a particle moving in a scalar potential (V) with an effective mass $\mu > 0$, which depends on the parameter γ [9]. When $\gamma = 1$, then $\mu = m$, the relativistic invariant mass of the particle; thus, Equation (1) coincides with the Schrödinger equation [3] [4]. Equation (1) is the Grave de Peralta (GP) equation when γ is the classical Lorentz factor of the special theory of relativity [5] [6] [7] [10]:

$$\gamma = \sqrt{1 + \frac{p^2}{m^2 c^2}}. \quad (2)$$

In Equations (1) and (2), c is the speed of the light in vacuum, \hbar is the reduced Plank constant, and $p = |\mathbf{p}|$, where \mathbf{p} is the particle's three-dimensional linear momentum. Another practical choice for the parameter γ was proposed by Poveda [8]:

$$\gamma = \langle \Psi | \hat{\gamma} | \Psi \rangle = \sqrt{1 + \frac{\langle \Psi | \hat{p}^2 | \Psi \rangle}{m^2 c^2}}, \text{ with } \hat{\gamma} = \sqrt{1 + \frac{\hat{p}^2}{m^2 c^2}}. \quad (3)$$

The quantum operators in Equations (1) and (3) are [1]-[9]:

$$i\hbar \frac{\partial}{\partial t} = \hat{H}' = \hat{H} - mc^2, \quad \hat{p} = -i\hbar \nabla, \quad \hat{K} = \frac{\hat{p}^2}{(\gamma+1)m} = -\frac{\hbar^2}{2\mu} \nabla^2. \quad (4)$$

In Equation (4), the energy operator (H') is associated to the total energy of the particle after discounting the energy associated to its mass (mc^2), and the kinetic energy operator (K) includes the correct relativistic relation with the linear momentum operator [5] [6] [7] [8]. Consequently, in the Poveda's approach, γ is the average value of the Lorentz factor in the quantum state Ψ [8]. Notably, Equation (1) reduces to well-known quantum wave equations in both the non-relativistic and ultra-relativistic limits. In the ultra-relativistic limit $p \gg mc$ ($\gamma \gg 1$), Equation (1) with $V = 0$ reduces to the following Weyl-like equation for a free spin-0 particle [11]:

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{p}c\Psi \Leftrightarrow \frac{1}{c} \frac{\partial}{\partial t} \Psi = -\nabla \Psi. \quad (5)$$

Previous works has focused in establishing the existing relationship between Equation (1) and the best-known relativistic quantum mechanics wave equations [5] [6] [7] [8] [9] [11]. Taking advantage of the formal similitude between the gGP and Schrödinger equations, Equation (1) have been solved, following the same mathematical procedures required for solving the same problems using the Schrödinger equation, for a free particle [7], confinement of a quantum particle in box [5] [6] [7] [8], reflection by a sharp quantum potential [5], tunnel effect [5], the harmonic oscillator [9], and the Hydrogen atom [12] [13]. This allowed the extension to the relativistic domain of well-known results previously obtained using the Schrödinger equation [3] [4].

In this work, the attention is focused on the relationship of Equation (1) with well-established theoretical methods. We show here, for the first time, how Equation (1) and the following equation [14]:

$$i\hbar \frac{\partial}{\partial t} \Psi = \frac{\hbar^2}{(\gamma+1)m} \nabla^2 \Psi + V\Psi \quad (6)$$

can be obtained by abduction using the Lagrange and Hamilton-Jacobi formalisms [1] [15] [16]. Equation (6) can be rewritten as Equation (1) but with $\mu < 0$. Equation (6) is a complementary gGP-like equation recently reported [14]. In contrast with Equation (1), Equation (6) only has solutions with negative kinetic energy values [14]. This work is then directed to explore the theoretical foundations of Equation (1) for any real value of μ .

The rest of this paper is organized in the following way. In Section 2, for self-reliance purposes, a summary is presented about how Equations (1) and (6) can be formally obtained. Then, in Sections 3 and 4, for the first time, we present how these equations can be obtained by abduction using Hamilton-Jacobi and Lagrange formalisms. Finally, the conclusions of this work are given in Section 5.

2. Poirier-Grave De Peralta Equations

How to obtain Equations (1) and (6), by applying a formal first quantization procedure on a Lorentz-invariant equation, which involves the energy-momentum 4-component vector (p^μ), has been discussed before [8] [14]. In short, we can start from the following Lorentz-covariant equation, which gives the module square of the energy-momentum 4-component vector of a free classical particle with mass m , energy E , and 3-component linear momentum \mathbf{p} [1]:

$$p^\mu p_\mu = g^{\mu\nu} p_\nu p_\mu = m^2 c^2. \quad (7)$$

In Equation (7), $g^{\mu\nu}$ is the metric tensor and p^μ and p_μ are the covariant and contravariant energy-momentum 4-component vector, respectively [1]:

$$p_\mu = \left\{ \frac{E}{c}, -\mathbf{p} \right\} = \left\{ \frac{E}{c}, -p_x, -p_x, -p_z \right\}, \quad p^\mu = \left\{ \frac{E}{c}, \mathbf{p} \right\} = \left\{ \frac{E}{c}, p_x, p_x, p_z \right\}. \quad (8)$$

The Lorentz-covariant Equation (7) can be transformed in the following Lo-

rentz-covariant quantum wave equation:

$$\hat{p}^\mu \hat{p}_\mu \Psi = g^{\mu\nu} \hat{p}_\nu \hat{p}_\mu \Psi \Leftrightarrow -\hbar^2 g^{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \Psi \Leftrightarrow \left(\frac{\hat{H}^2}{c^2} - \hat{p}^2 \right) \Phi = m^2 c^2 \Phi. \quad (9)$$

by making the following formal first-quantization substitution [1]:

$$E \rightarrow \hat{H} = i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow \hat{\mathbf{p}} = -i\hbar \nabla. \quad (10)$$

Equation (9) is the Klein-Gordon equation for a free spin-0 particle with mass m [1] [2]. We can proceed in the same way for obtaining the Klein-Gordon equation for a particle moving in a scalar potential V , but we should substitute E by $E - V$ and H by $H - V$ in Equations (8) and (9), respectively. We then obtain [1] [2]:

$$\left(g^{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} + \frac{m^2 c^2}{\hbar^2} \right) \Phi = 0 \Leftrightarrow \left[(\hat{H} - V)^2 - m^2 c^4 \right] \Phi = -c^2 \hbar^2 \nabla^2 \Phi. \quad (11)$$

Due to its relative importance [1] [2] [3] [4], we will limit the scope of this work to study the wave equations of a spin-0 particle with mass moving in a scalar potential. The general case including the presence of an external magnetic field will be discussed in a separated work. Equation (11) can be rewritten in the following way [8]:

$$\left[(\hat{H} - V) - mc^2 \right] \left[(\hat{H} - V) + mc^2 \right] \Phi = c^2 \hat{p}^2 \Phi. \quad (12)$$

Using the following identities [8] [9]:

$$\hat{K} = \frac{\hat{p}^2}{(\hat{\gamma} + 1)m} = -\frac{\hbar^2}{(\hat{\gamma} + 1)} \nabla^2 = (\hat{\gamma} - 1)mc^2 = \sqrt{\hat{p}^2 c^2 + m^2 c^2} - mc^2. \quad (13)$$

When $K = (E - V) - mc^2 > 0$, then $(E - V) + mc^2 = (\gamma + 1)mc^2$; therefore, Equation (12) can be rewritten as [5]:

$$\left[(\hat{H} - V) - mc^2 \right] \Phi = -\frac{\hbar^2}{(\hat{\gamma} + 1)} \nabla^2 \Phi. \quad (14)$$

However, when $K = (E - V) + mc^2 < 0$, then $(E - V) - mc^2 = -(\gamma + 1)mc^2$; therefore, Equation (12) can be rewritten as [14]:

$$\left[(\hat{H} - V) + mc^2 \right] \Phi = \frac{\hbar^2}{(\hat{\gamma} + 1)} \nabla^2 \Phi. \quad (15)$$

Consequently, Equation (12) reduces to:

$$i\hbar \frac{\partial}{\partial t} \Phi_\pm = \begin{cases} -\frac{\hbar^2}{(\hat{\gamma} + 1)} \nabla^2 \Phi_+ + V \Phi_+ + mc^2 \Phi_+, & \text{if } K > 0 \\ +\frac{\hbar^2}{(\hat{\gamma} + 1)} \nabla^2 \Phi_- + V \Phi_- - mc^2 \Phi_-, & \text{if } K < 0 \end{cases} \quad (16)$$

The Poirier-Grave de Peralta (PGP) equations are obtained from Equation (16) after introducing the wavefunctions Ψ_\pm in the following way [8]:

$$i\hbar \frac{\partial}{\partial t} \Psi_{\pm} = \begin{cases} -\frac{\hbar^2}{(\hat{\gamma}+1)} \nabla^2 \Psi_{+} + V \Psi_{+}, \text{ with } \Psi_{+} = \Phi_{+} e^{-\frac{i}{\hbar} mc^2 t} & \text{if } K > 0 \\ +\frac{\hbar^2}{(\hat{\gamma}+1)} \nabla^2 \Psi_{-} + V \Psi_{-}, \text{ with } \Psi_{-} = \Phi_{-} e^{+\frac{i}{\hbar} mc^2 t} & \text{if } K < 0 \end{cases} \quad (17)$$

Using Equation (13), Equation (17) can be rewritten as:

$$i\hbar \frac{\partial}{\partial t} \Psi_{\pm} = \begin{cases} +\sqrt{\hat{p}^2 c^2 + m^2 c^2} \Psi_{+} + V \Psi_{+} - mc^2 \Psi_{+}, \text{ with } \Psi_{+} = \Phi_{+} e^{-\frac{i}{\hbar} mc^2 t} & \text{if } K > 0 \\ -\sqrt{\hat{p}^2 c^2 + m^2 c^2} \Psi_{-} + V \Psi_{-} + mc^2 \Psi_{-}, \text{ with } \Psi_{-} = \Phi_{-} e^{+\frac{i}{\hbar} mc^2 t} & \text{if } K < 0 \end{cases} \quad (18)$$

The top equation in Equation (18) is related to the so called “spinless Salpeter” equation, a relativistic equation widely used to describe hadrons as bound states of the constituent quarks [17] [18] [19] [20]. Note the operator $i\hbar \partial/\partial t - mc^2$ in the top equation in Equation (16) is the operator H' in Equations (1) and (4). Different approximations may be used for substituting in Equation (17) the operator γ by a parameter γ [8] [9], thus transforming Equation (17) in Equation (1) for any real value of μ . For instance, we could use the Poveda’s approach that considers γ as the average value of the operator γ in the quantum state Ψ_{\pm} (Equation (3)). Alternatively, we could use the Grave de Peralta’s approach that considers γ as the classical Lorentz’s factor of special theory of relativity.

The energies $E'_{+} = E_{+} - mc^2$ and the wavefunctions Ψ_{+} , that can be obtained by solving the top equation in Equation (17), correspond to a spin-0 quantum particle with mass $m_p = m$, energies $E_p = E'_{+}$, wavefunctions $\Psi_p = \Psi_{+}$, and kinetic energies $K_p = K_{+} = E'_{+} - V > 0$ [5] [6] [7] [8] [9] [11] [12] [13]. Assuming that γ is a parameter, if the particle is a charged particle with charge $q_a = q$, and V is a potential produced by others charged particles surrounding the particle, then V change of sign when the sign of all the particles producing the potential change of sign (C-transformation). Consequently, the top equation in Equation (17) can be obtained from the bottom equation (and vice versa) by making the charge-parity-time (CPT) symmetry operations [1]: $t \rightarrow -t$ (T-transformation), $\mathbf{r} \rightarrow -\mathbf{r}$ (P-transformation), and $V \rightarrow -V$ (C-transformation); i.e. [14]:

$$\begin{aligned} i\hbar \frac{\partial}{\partial(-t)} \Psi_{-} &= \frac{\hbar^2}{(\gamma+1)} \nabla^2 \Psi_{-} + (-V) \Psi_{-} \\ \Leftrightarrow i\hbar \frac{\partial}{\partial t} \Psi_{+} &= -\frac{\hbar^2}{(\gamma+1)} \nabla^2 \Psi_{+} + V \Psi_{+}. \end{aligned} \quad (19)$$

In resemblance of the hole theory for the Klein-Gordon and Dirac equations [1] [14], this means that if the top equation in Equation (17) describes the states of a charged particle moving in a scalar potential V , which is produced by others charged particles surrounded the particle; then, the bottom equation in Equation (17) describes the states of the associated antiparticle moving in a scalar potential $-V$, which is produced by the corresponding charged antiparticles surrounded the antiparticle (matter-antimatter indistinguishability). This implies

that the energies $E'_- = E_- + mc^2$ and the wavefunctions Ψ_- , that can be obtained by solving the bottom equation in Equation (17), are related to the corresponding energies ($E_a = -E'_-$) and wavefunctions of the antiparticle associated to the spin-0 particle [14].

3. Hamilton-Jacobi Formulation of the gGP Equations

The relativistic Lagrangian corresponding to a classical particle of mass m moving in a scalar potential V with speed v is given by the following expression [15]:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - V, \text{ with } \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \sqrt{1 + \frac{p^2}{m^2 c^2}}. \quad (20)$$

The relativistic Lagrangian defined by Equation (20) is not written in a covariant form, but written in a preferred inertial frame, while V is a scalar potential depending only on position, which is related to a conservative force in the Lorentz inertial frame under consideration [15]. The corresponding particle's relativistic Hamiltonian, written in the preferred inertial frame, is then given by [15]:

$$H - V = v \frac{\partial \mathcal{L}}{\partial v} - \mathcal{L} = \sqrt{p^2 c^2 + m^2 c^4} = \gamma mc^2. \quad (21)$$

Therefore:

$$(H - V)^2 = p^2 c^2 + m^2 c^4. \quad (22)$$

Note that Equation (22) is not equivalent to Equation (21) because Equation (21) gives only one of the two possible solutions of Equation (22):

$$H - V = \begin{cases} v \frac{\partial \mathcal{L}}{\partial v} - \mathcal{L} = \sqrt{p^2 c^2 + m^2 c^4} = \gamma mc^2, & \text{if } H - V > mc^2 \\ -\left(v \frac{\partial \mathcal{L}}{\partial v} - \mathcal{L}\right) = -\sqrt{p^2 c^2 + m^2 c^4} = -\gamma mc^2, & \text{if } H - V < -mc^2 \end{cases} \quad (23)$$

While a classical particle only can have non-negative values of the kinetic energy ($H - V - mc^2$), both cases in Equation (23) are possible for quantum particles. The relativistic Hamilton-Jacobi equation for a classical particle is [16]:

$$(\nabla S)^2 - \frac{1}{c^2} \left[\frac{\partial S}{\partial t} + V \right]^2 + m^2 c^4 = 0. \quad (24)$$

It can be obtained from Equation (22) using the following relations [16] [17]:

$$\mathbf{p} = \nabla S, \quad H = -\frac{\partial S}{\partial t}. \quad (25)$$

In Equations (24) and (25), S is the action, which is defined by the following expression [16]:

$$S(r, t) = S_o + \int_{\delta(r_o, t_o)}^{(r, t)} \mathcal{L} dt. \quad (26)$$

The integral in Equation (26) should be evaluated along the extremal curve δ

joining the initial and final positions at the initial and final times; the final and initial positions and the final time should be considered variables while the initial position is fixed [16]. Equation (24) can be rewritten as:

$$\left[\left(\frac{\partial S}{\partial t} + V \right) + mc^2 \right] \left[- \left(\frac{\partial S}{\partial t} + V \right) + mc^2 \right] \Phi = c^2 (\nabla S)^2 \Phi. \quad (27)$$

The formal first quantization procedure described in Section 2 corresponds, in the Hamilton-Jacobi formalism, to the so-called Schrödinger *ansatz* [16]:

$$\begin{aligned} S(r, t) &= -i\hbar \ln[\Phi(r, t)] \\ \Rightarrow \frac{\partial S}{\partial t} \Phi &= -i\hbar \frac{\partial \Phi}{\partial t} = \hat{H} \Phi \quad \text{and} \quad \nabla S \Phi = -i\hbar \nabla \Phi = \hat{p} \Phi. \end{aligned} \quad (28)$$

This transform Equation (24) in Equation (11) and Equation (27) in Equation (12). As it was shown in the previous Section, from Equations (11) and (12) follows Equation (1). We have then found how to obtain Equation (1), for any real value of μ , from the relativistic Lagrangian corresponding to a classical particle of mass m moving in a scalar potential V (Equation (20)). A more heuristic analysis seems pertinent here. It has been showed above how to obtain the gGP equation (Equation (1)): from a proper Hamilton-Jacobi equation that enunciates a regularity in a classical wave's phase (Equations (24) and (27)) to an *ansatz* expression (Equation (28)) that allows the construction of an equation for the whole quantum wave (Equations (11) and (12)), and then the identification of the quantum operators involved. Perhaps engineer-minded people would identify this heuristic path as reverse engineering, something logicians prefer to call an abduction (the best possible explanation). In any case, the Schrödinger-like wave approach is a step back to continuousness, to a partial differential equation defined on the classical configuration space and abducted from the Hamilton-Jacobi framework that may offer a new way to approach complementarity, understanding it not only as a wave-particle duality but also as a tight-connected regularity via phase's consistency. Notice that the same Schrödinger's *ansatz* (Equation (28)) provides a unique bridge between classical mechanics and quantum mechanics independently of their relativistic or non-relativistic character.

4. Lagrange Formulation of the gGP Equations

Quantum mechanics equations can also be obtained from a Lagrange density that depends on the fields Ψ and Ψ^* [1]. In this Section, for clarity, we will denote the quantum fields corresponding to the \pm gGP equations as Ψ_+ and Ψ_- , thus they satisfy Equation (1) with $\mu_+ > 0$ and $\mu_- < 0$, respectively. Therefore, the quantum field Ψ_{\pm}^* satisfies the same equations but with i changed by $-i$. Assuming that γ is a parameter, it is easy to show that the Lagrange densities corresponding to the \pm gGP equations are given by the following expressions:

$$\mathcal{L}_+ = -\frac{\hbar^2}{2m} (\nabla \Psi_+) \cdot (\nabla \Psi_+^*) - \frac{(\gamma+1)}{2} \Psi_+ V \Psi_+^* - \frac{(\gamma+1)}{2} i\hbar \Psi_+ \frac{\partial \Psi_+^*}{\partial t}. \quad (29)$$

And:

$$\mathcal{L}_- = -\frac{\hbar^2}{2m}(\nabla\Psi_-) \cdot (\nabla\Psi_-^*) - \frac{(\gamma+1)}{2}\Psi_- V\Psi_-^* + \frac{(\gamma+1)}{2}i\hbar\Psi_-^* \frac{\partial\Psi_-}{\partial t}. \quad (30)$$

This means, the quantum fields Ψ_{\pm} and Ψ_{\pm}^* must satisfy the following Euler-Lagrange (field or wave) equations [1]:

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}_{\pm}}{\partial \left(\frac{\partial \Psi_{\pm}}{\partial t} \right)} \right] &= \frac{\partial \mathcal{L}_{\pm}}{\partial \Psi_{\pm}} - \nabla \left[\frac{\partial \mathcal{L}_{\pm}}{\partial \nabla \Psi_{\pm}} \right], \\ \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}_{\pm}}{\partial \left(\frac{\partial \Psi_{\pm}^*}{\partial t} \right)} \right] &= \frac{\partial \mathcal{L}_{\pm}}{\partial \Psi_{\pm}^*} - \nabla \left[\frac{\partial \mathcal{L}_{\pm}}{\partial \nabla \Psi_{\pm}^*} \right]. \end{aligned} \quad (31)$$

A straight substitution of Equations (29) and (30) in Equation (31) demonstrates that Equations (29) and (30) gives the correct Lagrange densities corresponding to the \pm gGP equations. For instance, substituting Equation (29) in Equation (31), we obtain:

$$\frac{\partial \mathcal{L}_+}{\partial \left(\frac{\partial \Psi_+}{\partial t} \right)} = 0 \Rightarrow \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}_+}{\partial \left(\frac{\partial \Psi_+}{\partial t} \right)} \right] = 0, \quad (32)$$

$$\begin{aligned} \frac{\partial \mathcal{L}_+}{\partial \left(\frac{\partial \Psi_+^*}{\partial t} \right)} &= -\frac{(\gamma+1)}{2}i\hbar\Psi_+ \Rightarrow \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}_+}{\partial \left(\frac{\partial \Psi_+^*}{\partial t} \right)} \right] = -\frac{(\gamma+1)}{2}i\hbar \frac{\partial \Psi_+}{\partial t}, \\ \frac{\partial \mathcal{L}_+}{\partial \Psi_+} &= -\frac{(\gamma+1)}{2}i\hbar \frac{\partial \Psi_+^*}{\partial t} - \frac{(\gamma+1)}{2}V\Psi_+^*, \\ \frac{\partial \mathcal{L}_+}{\partial \Psi_+^*} &= -\frac{(\gamma+1)}{2}V\Psi_+. \end{aligned} \quad (33)$$

And:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \nabla \Psi_+} &= -\frac{\hbar^2}{2m}\nabla\Psi_+^* \Rightarrow \nabla \left[\frac{\partial \mathcal{L}}{\partial \nabla \Psi_+} \right] = -\frac{\hbar^2}{2m}\nabla^2\Psi_+^*, \\ \frac{\partial \mathcal{L}}{\partial \nabla \Psi_+^*} &= -\frac{\hbar^2}{2m}\nabla\Psi_+ \Rightarrow \nabla \left[\frac{\partial \mathcal{L}}{\partial \nabla \Psi_+^*} \right] = -\frac{\hbar^2}{2m}\nabla^2\Psi_+. \end{aligned} \quad (34)$$

Therefore:

$$\begin{aligned} -i\hbar \frac{\partial \Psi_+^*}{\partial t} &= -\frac{\hbar^2}{(\gamma+1)m}\nabla^2\Psi_+^* + V\Psi_+^*, \\ i\hbar \frac{\partial \Psi_+}{\partial t} &= -\frac{\hbar^2}{(\gamma+1)m}\nabla^2\Psi_+ + V\Psi_+. \end{aligned} \quad (35)$$

As expected, in the non-relativistic limit the Lagrange density corresponding

to the +gGP (Equation (29) with $\gamma = 1$) is also the Lagrange density corresponding to the Schrödinger equation [1].

5. Conclusion

We presented how to obtain a pair of relativistic quantum mechanics equations by abduction from the Hamilton-Jacobi and Lagrange formalisms. In contrast with Klein-Gordon and others relativistic quantum mechanics equations, the +gGP equation (Equation (1) with $\mu > 0$) has only solutions with positive kinetics energies, while the -gGP equation (Equation (1) with $\mu < 0$) has only solutions with negative kinetics energies. Therefore, solving the +gGP equation allows for finding the wavefunctions $\Psi_p = \Psi_+$ and energies ($E_p = E'_+$) of a spin-0 particle of mass m , which is moving in a scalar potential at relativistic speeds. Complementarily, solving the -gGP equation allows for finding information about the energies ($E_a = -E'_-$) and wavefunctions corresponding to the associated antiparticle.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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